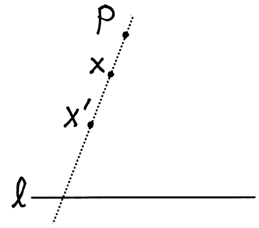
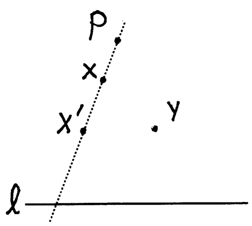
Projective Geometry Puzzle #1

Homology

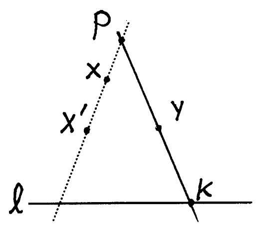


Background: A homology allows us to move any chosen point on the plane to a new point or location. This is specified by three factors (shown on the right): a center (P), an axis (ℓ), and a given pair of corresponding points (X and X') on ℓ.

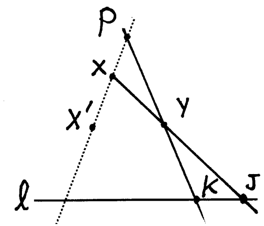
Our task is to use these given conditions to determine where any chosen point, Y, will end up, which is the desired point, Y'. The below example will help to clarify matters.



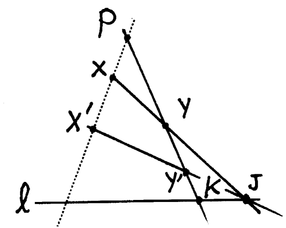
Choose any point Y.



Draw a line through P and Y to find K on ℓ.



Draw a line through X and Y to find J on ℓ.



Draw a line through J and X' to find Y' on line PY.

Problems:

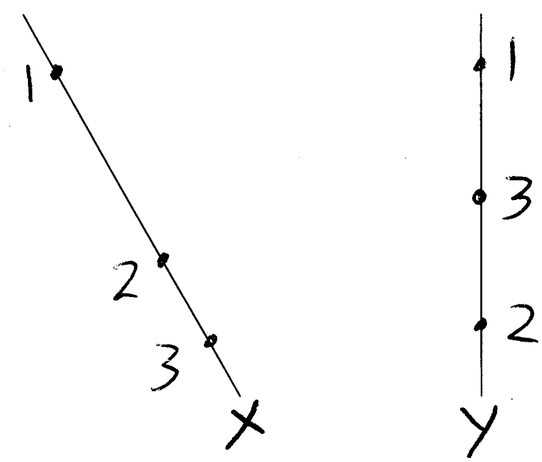
1. Experiment with several other locations for the initial chosen point, Y.
2. Where must Y be placed so that it is self-corresponding (i.e., Y' ends up exactly at Y)?
3. Draw a random line, n, and find out where its corresponding line, n', ends up. (Hint: use two points on line n.)
4. What must always be true of any line, n, and its corresponding line, n' ?
5. As a point, Y, moves along a random line, n, what happens to the corresponding point, Y', as point Y moves further and further off the page?
6. If line n moves further and further off the page, what happens to its corresponding line, n' ?

Projective Geometry Puzzle #2

Fundamental Theorem of Projective Geometry.

Given three (random) points (labeled 1, 2, 3) on line X, and three points (1, 2, 3) on line Y, find a projectivity that projects these 3 points from line X to line Y.

In other words, imagine that someone does a projectivity – without you observing – by starting with three points on line X, then perspects those points through P (the point of perspectivity) onto line ℓ, then perspects those points through Q onto line Y. After doing all of this, the person then erases the intermediate points P and Q, and the intermediate line ℓ. Your task is to determine where P, Q, and ℓ could possibly be (and there are multiple possibilities!) such that the three points start on line X (as given) and end up on line Y (as given). Try the below example!



Projective Geometry Puzzle #3

The Theorem of Pappus in Point Geometry

Background

- With (normal) plane 2-D geometry, we work with a singular given plane (our page), which has infinitely many lines and points lying on it. If we take the previous sentence and swap the terms “plane” and “point”, we get: With point 2-D geometry, we work with a singular given point, which has infinitely many lines and planes passing through it.
- In plane geometry, we are only working with points and lines – the only plane is the one we are working on. In point geometry, we are only working with planes and lines – the only point is the one we are working on.
- Any projective geometry theorem in plane geometry (with points and lines on the given plane) can be translated so that it also works in point geometry (with planes and lines through the given point).
- Here are some tips to help “construct” the Theorem of Pappus in point geometry:
    - The difficulty is that we cannot easily draw the Theorem of Pappus in point geometry because we are working (in a Euclidean sense) in 3-D space. So instead of constructing a drawing on paper, we will need to just work with our imagination.
    - We normally work with plane geometry by drawing points and lines on the page. We can translate the Theorem of Pappus into point geometry by considering the steps required to complete a drawing (in plane geometry) and then rework the steps by swapping the terms “points” and “planes”.
    - In plane geometry, the intersection of any two lines is a point that must fall on the given plane (your page), and the joining line of any two points must be on the given plane. Likewise, in point geometry, the common plane to any two lines must pass through the given point, and the line of intersection of any two planes is a line that must pass through the given point.

The Questions

1. Describe as accurately as possible what happens to the Theorem of Pappus in point geometry?
2. Describe as accurately as possible what happens to the Dual of Pappus in point geometry?

Projective Geometry Puzzle #4

The Tangent Puzzle

Part 1: Without drawing any tangent lines, find a method for constructing the polar line to any given point with respect to the polarity conic. This method should work whether the given point is inside or outside the polarity conic, and it should work for a non-circular polarity conic. Also, find a method for constructing the polar of a line.

Part 2: Find a method to construct a line tangent to a conic through a given point outside the conic.

Part 3: Find a method to construct a line tangent to a conic through a given point on the conic.

The Key Theorem (for each of the above puzzles): **The Inscribed Quadrangle Theorem**:

"With any quadrangle inscribed in a conic, its diagonal triangle is self-polar." (Coxeter, p75)

- Construction: Choose a quadrangle (4 points) such that it is inscribed on the conic. Draw the 6 lines connecting these 4 points, and then locate the three new points of intersection, which form the *diagonal triangle*. To say that this diagonal triangle is self-polar means that each point of the triangle is polar to the opposite side, etc.

Solutions to Projective Geometry Puzzles

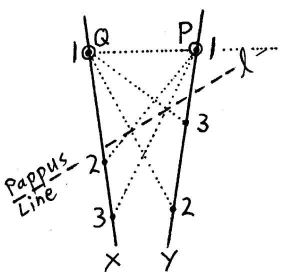
<u>Solution to Projective Geometry Puzzle #1</u>

2) Y has to be anywhere on line ℓ;
4) lines n and n' must meet on line ℓ;
5) point Y' moves through infinity and comes back from the other side;
6) n' becomes closer and closer to becoming parallel to line ℓ.

<u>Solution to Projective Geometry Puzzle #2</u>

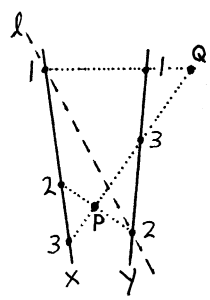
I know of three possible methods to solve this problem.

Method #1



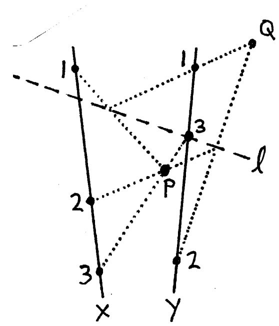
The Pappus line is used as line ℓ, and points P and Q coincide with a pair of corresponding points.

Method #2



Line ℓ is chosen so that it goes through two non-corresponding points. This then determines where P and Q must be placed.

Method #3



Choose one of the six points. Draw line ℓ through this chosen point. Also, place P or Q (you'll have to think about which one) somewhere on the line that connects the chosen point with its corresponding point. This determines where the other point of perspectivity (P or Q) must be. (Note: with the above drawing, it is coincidental that point Q looks to be on the extended line that joins the two points labeled "3".)

<u>Solution to Projective Geometry Puzzle #3</u>

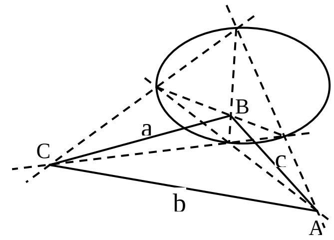
The Theorem of Pappus and its Dual in Plane Geometry

- Theorem of Pappus: *"On any plane, given any three points* (A, B, C) *on one line and another three points* (A', B', C') *on another line, the three corresponding pairs of joining lines* (AB' & A'B, AC' & A'C, BC' & B'C) *meet in points of intersection that are collinear."* (We start with two lines, each line with three points on it, and the end result is a line with three points on it.)
- Dual of Pappus: *"On any plane, given any three lines* (a, b, c) *on one point and another three lines* (a', b', c') *on another point, the three corresponding pairs of points of intersection* (ab' & a'b, ac' & a'c, bc' & b'c) *are joined by lines that are concurrent."* (We start with two points, each point with three lines passing through it, and the end result is a point with three lines passing through it.)

The Theorem of Pappus and its Dual in Point Geometry

- Theorem of Pappus: *"On any point, given any three planes* (α, β, γ) *on one line and another three planes* (α', β', γ') *on another line, the three corresponding pairs of lines of intersection* (αβ' & α'β, αγ' & α'γ, βγ' & β'γ) *form coaxial planes."* (We start with two lines, each line with three planes on it, and the end result is a line with three planes on it.)
- Dual of Pappus: *"On any point, given any three lines* (a, b, c) *on one plane and another three lines* (a', b', c') *on another plane, the three corresponding pairs of planes* (ab' & a'b, ac' & a'c, bc' & b'c) *have intersections that are coplanar lines."* (We start with two planes, each plane with three lines on it, and the end result is a plane with three lines on it.)

Solution to Projective Geometry Puzzle #4



- **Part 1** (Finding the polar of a point or line without using tangent lines)

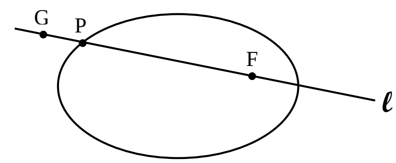
  To construct the polar of a point (A), simply draw two lines through the point that intersect the conic. This yields four points of intersection on the conic. Use these four points to construct the diagonal triangle (ABC). The Inscribed Quadrangle Theorem says that this triangle is self-polar. Therefore line a is polar to point A. To construct the polar of a line, simply choose two points on the line and follow the same above process.

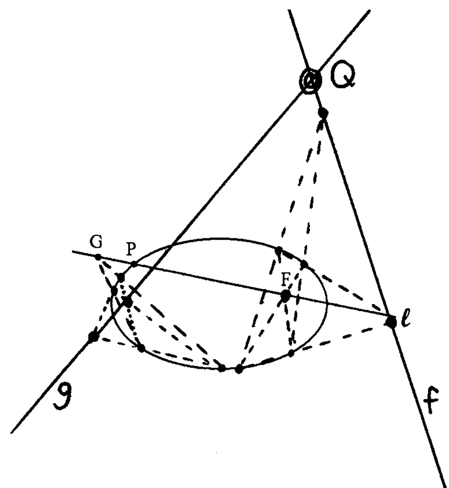
- **Part 2**: (Drawing a tangent line to a conic through a given point, A, <u>outside</u> the conic)

  To draw a tangent line from point A, simply find the polar line (a) using the above method. The two points of tangency are the places where line a (extended) crosses the conic.

- **Part 3**: (Drawing a tangent line to a conic through a point, P, on the conic)

  1. Draw any secant line (***ℓ***) through point P and choose any two points (F, G) on line ***ℓ***.





  2. Our primary objective is to find the polar to line ***ℓ*** without drawing any tangents. We can do this by taking points F and G and finding their polar lines (f and g), by using the method from Part 1. The polar to line ***ℓ*** is point Q, which is where lines f and g meet.

  3. The desired tangent line is the line connecting P and Q.