Lesson Plans for Projective Geometry

11th Grade Main Lesson (last updated December 2020)

Overview

In many ways projective geometry – a subject which is unique to the Waldorf math curriculum – is the climax of the students' multi-year study of geometry in a Waldorf school. The thinking involved is both demanding and creative. It dramatically alters their previous experience and notion of geometry.

Projective geometry attempts to answer the question: "What laws of geometry are still valid that have nothing to do with measurement?" We start, however, with a philosophical debate. Under some circumstances, it appears that two parallel lines meet. For example, artists during the Renaissance noticed that two lines, which are known to be parallel, actually meet in the drawing. Historically, it took a couple hundred more years before people dared to question Euclid's fifth postulate, which essentially states that two parallel lines never meet. We then decide to work – perhaps somewhat skeptically at first – with the assumption that two parallel lines meet at infinity. What happens then? This leads us to investigate many different theorems in projective geometry, including theorems from Pappus, Desargues, Pascal and Brianchon. The topics get more sophisticated during the second half of the course as we study the principle of duality, line-wise conics, and conclude with an in-depth study of polarity.

Notes for the Teacher

- *Block Test?* I question the value of having a test at the end of a projective geometry block. If I were to give such a block test, then I would design it so that the students would feel good about what they have learned.
- *Grades*. I prefer not to grade this block – how do you grade their imagination?
- *Drawings*. The purpose of PG is <u>not</u> to just make pretty drawings. It is important for the students to understand deeply what the drawings represent. The process in the students' imagination is more important than the finished drawing. I want to encourage the students to experiment; not every drawing has to be in "perfect" finished form. Although I am not fond of having students display their work (people can't get much of an impression of what PG is by looking at a PG drawing), I do often have the students do a more complicated drawing, such as the polarity of a curve, as a "final project".
- *Projective Geometry Puzzles*. For those students who need an extra challenge, there are separate "Projective Geometry Puzzles" problems that can be downloaded from our website.

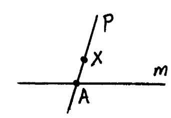
Day #1

- Hand out sheet on course expectations, business stuff, (e.g. being on time), etc.
- <u>What is "main stream" geometry</u>?
  - Proofs & logic, formulas (e.g. area, volume), measurement, Coordinate geometry, etc.
- <u>What is "pure" geometry?</u>
  - Geometric drawing (spirals, forms in movement, etc.), stereometry, loci, descriptive geo, projective.
- <u>What is Projective Geometry?</u>
  - We will not define it, but rather, slowly over time we will build up a picture that characterizes projective geometry. During this time, your understanding of what projective geometry is, will become altered.
  - Characterizing (not defining) PG: Lawrence Edwards' subtitle: "An approach to the secrets of space from the standpoint of artistic and imaginative thought".
  - You will be transcending what most people know about geometry.
- <u>The Complete Triangle in Movement</u>
  - Be clear that the "complete triangle" (or "projective triangle") divides the whole plane into 4 regions. Three of the regions "pass through infinity".
  - Have students do the drawing. Tell them that there are 12 stages. Show them the 12 stages on the board, with the first one colored in, with each of the four regions in a different color. They have to think about how to color in each region. It's harder than you think!
  - Ask: which region is bounded on all sides? (Ans: All four regions!)
  - A key question should be: "What happens to the apex of the triangle (imagine it to be blue) when the moving lines are parallel?" This could be quite a debate.
  - The only thing that makes it work is if the two parallel lines meet at one point (in both directions) at ∞.
- <u>Quadrangle and Quadrilateral</u> (if there is time)
  - Law: The four given lines of a quadrilateral define six points and then three lines (diagonal triangle).
  - Law: The four given points of a quadrangle define six lines and then three points (diagonal triangle).
- <u>Review of Conics (from earlier in the year, or from 8th grade?)</u>:
  - Section a clay cone. Explain how a hyperbola is <u>one</u> continuous curve, and the parabola is the magical instant between an ellipse (infinitely stretched out) and a hyperbola (one branch swallowed by ∞).
- <u>Main Lesson Book</u>: Finish Drawing on "The Complete Triangle in Movement"

Day #2

- Review: The daily standard question: "What new and interesting idea did you walk away with yesterday?"
- Characterizing PG:
  - Arthur Cayley (1859): "PG is all geometry"
  - Two points divide the line into two segments.
  - Many of the thoughts cannot really be understood in a "normal" sense. We will be studying many mysterious things. Make an analogy to not really understanding (completely) electricity.
    - Don't feel badly if you don't think you understand this; what does it mean to "understand", anyway?
  - With PG, "geometry as know it" gets turned on its head. For example:
    - No such thing as a point being "inside" a triangle;
    - We can no longer say that the angles in a triangle add to 180°.
    - In fact, the whole notion of a triangle is different…
  - Regarding The *Complete Triangle in Movement*: In PG, we don't like to have exceptions if we can avoid them. It is "neater" to say that the triangle's apex is always there – with no exception. When the lines are parallel, this means that the apex is infinitely far away in both directions.

- *Kepler's Propeller*. With the drawing shown here, the two lines intersect at point A. Now imagine that line p rotates clockwise like a propeller, and as it rotates, the point of intersection (A), moves along to the left.



  - **The big question**: Where is point A at the instant when the two lines are parallel?
    - Option #1: There is no point – the point disappears for an instant.
    - Option #2: There are two points – one point infinitely far in each direction.
    - Option #3: There is one point –infinitely far in both directions at the same time.

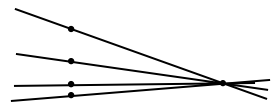
- Theorem of Pappus
  - Do not talk about an ordered hexagon yet. The statement for the drawing waits until tomorrow.
  - Simply label points A, B, C on one line and A', B', C' on the other.
  - Have each pair of corresponding joining lines in a different color (i.e., AB' and A'B are both blue, etc.). In this way, one of the intersecting points is where two green lines meet, one is where two orange lines meet, and one is where two blue lines meet.
  - These three points of intersection fall on one line – the "Pappus Line".
  - Have the students play with the drawings by moving the lines and points.
  - Try to take care that the points of intersection all fall on the page.
  - (If time allows) In groups…
    - Create a Pappus drawing where one pairs of corresponding lines (e.g., AB' and A'B) is parallel.
    - Create a Pappus drawing where two pairs of lines are parallel. What do you notice? (Answer: if two pairs of lines are parallel, then the third pair must also must be parallel.)
    - What happens if all three pairs of lines are parallel? (Answer: something strange!)
- Types of hexagons:
  - Hexangle: 6 points given, which define 15 lines and then 45 points.
  - Hexalateral: 6 lines given, which define 15 points and then 45 lines.
  - (Ordered) Hexagon: 6 points (or lines) given with their order, define 6 lines (or points) and their order.
  - Practice drawing Euclidean hexagons, then do projective ordered hexagons by drawing the connected line segments first, labeling the order, and then extending the line segments.
  - With an ordered hexagon, it is perhaps easiest to start with 6 points on the page (no three of which are collinear), label them in some order, and then connect the dots.
    - In this case 6 points correspond to 6 lines.
    - Color opposite sides of the ordered hexagon to have the same color (total of three colors).
- The Artist's Dilemma (Mention briefly – more to come later in the week.)
  - If you take a photo, does every point in the photo exist in reality?
  - This leads to the mystery of the drawing of the railroad tracks meeting at the horizon.
- (Only for those who need an extra challenge – see website) *Projective Geometry Puzzle #1 – Homology*.
- Hanging Questions:
  1) What can we do about the Artist's Dilemma?
  2) What happens with Th. of Pappus when one pair of corresponding lines is parallel? When two pairs of corresponding lines are parallel? When all three pairs of corresponding lines are parallel?
  3) What is the intersection of two parallel planes?
- Main Lesson Book: Finish at least two drawings on Pappus.

Day #3

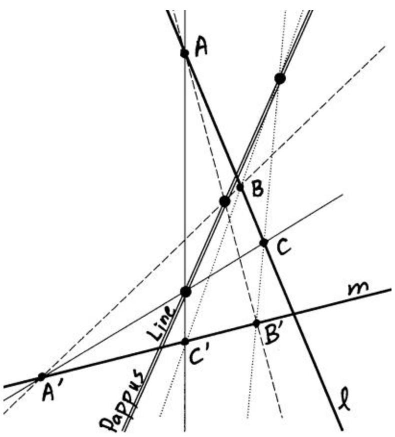
Review: The daily standard question: “What new and interesting idea did you walk away with yesterday?”

- Kepler’s propeller. We will go with option #3.
- **Our “postulate”: Two parallel lines meet at one point at infinity.**
- Have drawings on the board before class that review what was done yesterday.
- Review hanging questions from yesterday. This leads to the *Line at Infinity* (See below).

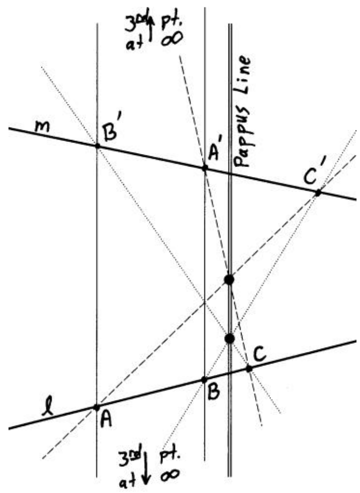
- Use the drawing at the right to show that three (or more) parallel lines meet at one point! Do this by imagining that the point of intersection moves to the right as the lines rotate about the other fixed points (that are vertical from one another).



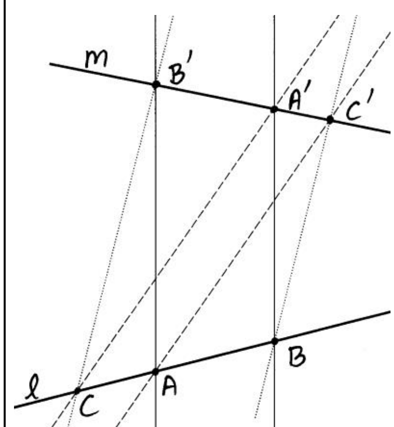
- Theorem of Pappus (continued).
  - *Play with Infinity!* Encourage the students to be adventurous and try some of the following:
    - Arrange it so that one of the intersection points is at infinity (see drawing #2, below).
    - Arrange it so that all three intersection points are at infinity (see drawing #3, below).
    - Start by placing A' at infinity, or A and A' at infinity, or A and B' at infinity (see #4 & #5).
    - Put the whole of line ***ℓ*** at infinity, with points A, B, C in three different directions (see #6).



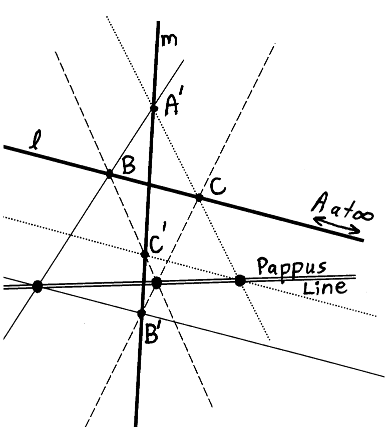
#1: “Normal” drawing of the Theorem of Pappus with all points on the page.



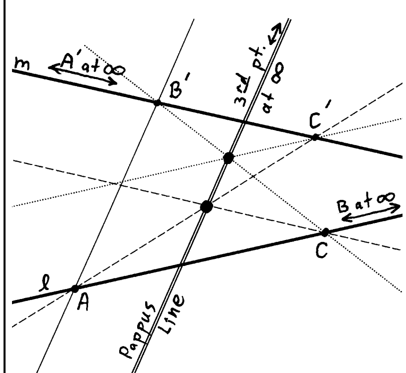
#2: Drawing of the Theorem of Pappus with the third intersecting point at infinity, but the Pappus line is still on the page.



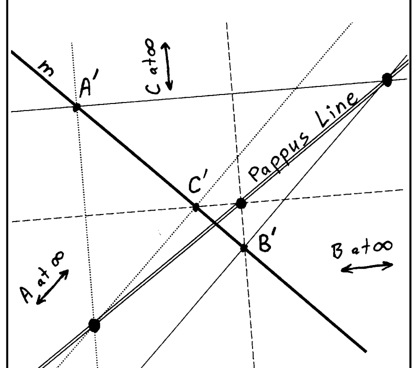
#3: Drawing of the Theorem of Pappus with all three intersecting points at infinity in different directions. The Pappus line is the line at infinity!



#4: Drawing of the Theorem of Pappus starting with A at infinity. Note that lines AB' and AC' must be parallel to line ***ℓ***.



#5: Drawing of the Theorem of Pappus starting with A' and B at infinity. A'B is then the line at infinity.



#6: Drawing of the Theorem of Pappus starting with the whole of line ***ℓ*** at infinity. The points A, B, C are in three different directions.

- Write up a statement of the Theorem of Pappus (we will give an alternate one later):

  Given any three points (A, B, C) *on one line and another three points* (A', B', C') *on another line, the three corresponding pairs of lines* (AB' & A'B, AC' & A'C, BC' & B'C) *meet in collinear points.*
- The Line at Infinity.
  - Go over hanging question #3 from yesterday (What is the intersection of two parallel planes?).
  - Arrive at it by doing the same as Kepler's propeller, but in two dimensions. Therefore, we have the floor, and a second plane pivoting on a point above the floor. As the above plane pivots, imagine a blue line on the floor that is moving. The obvious question is: where is the blue line at the instant the two planes become parallel?
  - That blue line could have been pushed "toward infinity" in any direction, but when the planes become parallel, the line ends up in the same place.
  - If two points are infinitely far away, then the line that joins them is the line at infinity.
  - Regarding the line at infinity…
    - Two parallel planes meet at this line.
    - It is incorrect (although tempting) to think of *line at infinity* as the line that (like a circle) "bounds" the plane. But, it is not a boundary –you can "go past" it. It is not a circle. The line at infinity is straight even though we have the illusion that it surrounds us!
    - The *line at infinity* is a collection of all of the points on a given plane that are infinitely far away.
    - Show how you can point to each one of the points on the *line at infinity* – with our arms, of course, pointing in both directions!
- Main Lesson Book: At least 3 drawings of the Theorem of Pappus involving playing with infinity.

Day #4

Review: The daily standard question: "What new and interesting idea did you walk away with yesterday?"

- Perhaps it is easier to think "Two parallel lines have one point in common that is infinitely far away."
- Important for students: If you have had a hard time believing the concept that two parallel lines meet at infinity, then try not to let this block your way from moving forward. For the remainder of the course, we will see several times where this strange thought will help us to do a drawing. You don't have to believe it – you just have to be open to using it; in that way you will learn some surprising things.
- Drawings from yesterday: Have drawings on the board before class that review what was done yesterday, especially including drawing that involve playing with infinity.

- Theorem of Pascal.
  - Start with a circle. Label any 6 points. Draw colored lines connecting points: 1→2 green, 2→3 orange, 3→4 purple, 4→5 green, 5→6 orange, 6→1 purple. Circle points of intersection of like-colored lines.
  - Pascal's Theorem works not just with a circle, but with any conic!! Students should try with all kinds of conics – circles, ellipses, parabolas, hyperbolas.
- Main Lesson Book: Drawings of the Theorem of Pascal.
- Characterizations:
  - Several PG theorems were discovered long before PG was known, such as Theorem of Pappus.
  - Two points divide the line into two segments.
  - No such thing as "betweenness". With 3 points on a line (A, B, C) I can get from A to C either by passing B, or not.
  - Quote for the day: "Euclidean geometry is the geometry of touch; PG is the geometry of sight."
  - In PG, we like to have Theorems that are as general as possible, so that it is true for all possible cases, without exceptions. An example of a PG postulate is "Any two co-planar lines meet at one point."

- Theorem of Pascal.
  - Start with a circle. Label any 6 points. Draw colored lines connecting points: 1→2 green, 2→3 orange, 3→4 purple, 4→5 green, 5→6 orange, 6→1 purple. Circle points of intersection of like-colored lines.
  - Pascal's Theorem works not just with a circle, but with any conic!! Students should try with all kinds of conics – circles, ellipses, parabolas, hyperbolas. Be sure to play with infinity!
- Hanging Question:
  - Imagine someone to be at sea at the top of a ship's mast looking through a telescope at the horizon. You are looking at them from some distance, and see that the telescope is pointed slightly downward. Then imagine that the earth grows until it is a perfect flat plane. How does the angle of the telescope change?
- Main Lesson Book: Five drawings of the Theorem of Pascal, three of which play with infinity.

Day #5

Review:

- Two Statements to write on the board:
  - Every line has one infinitely distant point, and we can get there by going in either direction.
  - Any plane has one infinitely distant line, and from any point on the plane we can "point" to a point on that line by pointing in any direction (with our arms, of course, pointing in both directions!).
- Another weird thing about parallel lines: Draw two horizontal parallel lines on the board. If the top one moves up by a foot (or by a mile!) it doesn't change the point where the two lines meet.
- Give statement for Pascal: *If the vertices of an ordered hexagon lie on a conic, then the three pairs of opposite sides meet in collinear points.*

Drawings from yesterday: Have drawings on the board before class that review what was done yesterday, especially including drawing that involve playing with infinity.

- Theorem of Pappus – revisited
  - Look at Pappus in a new way now – as a theorem having to do with an ordered hexagon.
    - Instead of labeling the points as before, we will now label them in order 1,2,3,4,5,6, so that the odd-numbered points are on one line, and the even-numbered points are on the other line. The ordered hexagon is then seen to bounce back and forth between the two lines as you move in order along the hexagon.
  - *A new statement for the Theorem of Pappus…*
    - Review our first statement of the Theorem of Pappus.
    - Actually, the Theorem of Pappus can be seen as a statement about a hexagon.

      If the vertices of an ordered hexagon lie alternately on two lines, then the three pairs of opposite sides meet in collinear points.

      (I don't think that this means the Theorem of Pappus is a special case of the Theorem of Pascal, where the two lines of Pappus are, in effect, a degenerate hyperbola. If this were true, then it would follow that the dual of Pappus would be a special case of Brianchon, and the two initial points of the dual of Pappus would have to be a degenerate case of a conic, but that doesn't seem possible.)
- Intro to Duality. (first formulated as a general principle of PG by Gergonne in 1826.)
  - Mention that any statement in PG can be stated also in its dual form.
  - This is one of the most amazing principles of PG: every theorem in PG is also true in its dual form. We only need to switch the words (and ideas) of "point" and "line".
  - Practice by taking the dual of various statements, such as:
    - Three points are on a line. A line passes through each of these points.
    - (Quadrilateral) 4 points, no 3 of which are collinear, define 6 lines and 3 new points.
    - The Theorem of Pascal's statement.
  - Two key postulates of PG: (1) Any two points have a common line; (2) Any two (coplanar) lines have a common point.
- Theorem of Brianchon
  - About 200 years after Pascal discovered his theorem, Brianchon discovered that the dual was also true. Let the students figure this out.
  - Be sure to outline the sides of the ordered hexagon in black ink; opposite vertices have the same color.
  - Try with all kinds of conics – circles, ellipses, parabolas, hyperbolas. Be sure to play with infinity!
- Perspectivity (Perspective from a point only) Give basic idea in this order:
  - A perspectivity of a triangle from one plane to another.
  - Renaissance artists held their head in place, and closing one eye perspected the scene onto their canvas.
  - Demonstrate this by imagining a triangle on the ground outside view through a window. Imagine having laser beams shoot from your eyes to each vertex of the triangle. Use a real marker to mark each point on the glass.
  - We can now say that the two triangles are in perspective because the three lines (laser beams) connecting corresponding points all meet in one point.
  - Ask: How can you tell if two figures, each on different plane, are in perspective, or not? Answer: line up corresponding points. If all of the lines are concurrent, then the answer is "yes".
- Hanging Question:
  - First review yesterday's hanging question. The person at the top of the ship's mast would have his arm parallel to the plane. In fact the horizon line would only be lifted a couple of degrees. If another person was somehow a mile above the ship's mast, then he would also have his arm parallel to the plane.
  - New question: Imagine taking a photo of a parabola such that the entire parabola is in view, the vertex is closest to the camera, and the parabola is "running" toward the horizon. What would it look like?
- Main Lesson Book: Write your essay for the week.

Day #6

Review:

- The daily standard question: "What new and interesting idea did you walk away with yesterday?"
- Yesterday's hanging question: A parabola looks like an ellipse tangent to the horizon. This is because a parabola is tangent to the line at infinity. As the parabola goes toward the horizon, its "sides" look like they come together, just as the sides of a train track look like they get closer together.
- How to find opposite sides of an ordered hexagon (have a couple of drawings on the board before class).
- *Duality*. Put these statements on the board:
  - Two (coplanar) lines meet in one point → Two points are joined by one line.
  - Two objects are perspective from a point if corresponding points are joined by concurrent lines. → Two objects are perspective from a line if corresponding lines meet in collinear points.
- A photo is a 3-D perspectivity from one plane to another. The aperature is the point of perspectivity.

Drawings from yesterday: Have drawings on the board before class that review what was done yesterday, especially including drawing that involve playing with infinity.

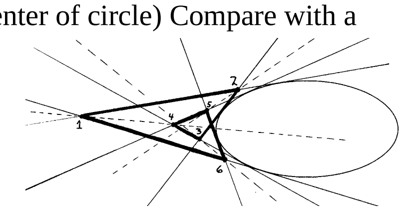
- Infinity and Dimensions.
  - On a plane (2D), any two parallel lines (1D) meet at one infinitely distant point (0D); every line (1D) includes one point (0D) at infinity.
  - In space (3D), any two parallel planes (2D) meet at one infinitely distant line (1D); every plane (2D) includes one line (1D) at infinity.
  - In 4D hyperspace, and two parallel 3D spaces meet at one infinitely distant plane (2D); every 3D space includes one plane (2D) at infinity.
- Excellent Exercise: There are two parallel lines that perpendicularly come out of your paper – one at the top right corner and the other at the top left corner. The point of perspectivity, P, is located about two inches above the center point of the page. Perspect the two parallel lines (by drawing them) through P onto the page.
  - Additionally, imagine that a red and blue point are moving side by side along the two parallel lines. At the same time, a corresponding red and blue point are moving along the (perspected) lines on the page.
    - Where are the points on the page when the points on the parallel lines are at infinity?
    - Where are the points on the page when the points on the parallel lines are below the page's plane?
    - Where are the points on the parallel lines when the points on the page are at infinity?
    - Where are the points on the page when the points on the parallel lines are between 0 and 2 inches above the page's plane?
- The idea of a 2-D perspectivity.
  - *Two polygons that are in perspective*. Give an example of two co-planar triangles that aren't in perspective, and two that are.
  - *A perspectivity of points on a line*. Show how a perspectivity can be done taking 5 points on one line to another line.
- Pascal and Brianchon. Put six drawings on the board.

  Theorem of Pascal

  1) Choose six points on a circle, and then number them in clockwise order.
  2) Redo the same above drawing with the same six points on a circle, but number them differently.
  3) Regular hex inscribed in a circle (Resulting line is at ∞.)

  Theorem of Brianchon (dual of Pascal)

  4) Choose six lines tangent to a circle, and then number them in clockwise order.
  5) Redo the same above drawing with the same six lines on a circle, but number them differently.
  6) Regular hex circumscribed about a circle (Resulting point is at center of circle) Compare with a regular hex for Theorem of Pascal.

  - Give statements for Brianchon:

    If the sides of an ordered hexagon are tangent to a conic, then the three pairs of opposite vertices can be joined by concurrent lines.
  - Brianchon's Theorem works for any 6 lines tangent to the conic. These lines don't have to "go around" the conic. (See drawing.)
  - Brianchon on a parabola: You can use the line at infinity to exactly construct a drawing! Similar to the above, we can simply draw five lines tangent to the parabola, and then use the line at infinity as the sixth line. Label these six lines in any order. The result should work just fine!

- The Dual of Pappus.
  - It is best for the students to figure this out for themselves.
  - If you get stuck in trying to do a dual drawing, you can simply write down each of the steps that were required to do the drawing, and then dualize each step.
  - Do both drawings on different sides of the board in front of the class, or write down the step-by-step directions for Pappus and have the students figure out the directions of the dual.

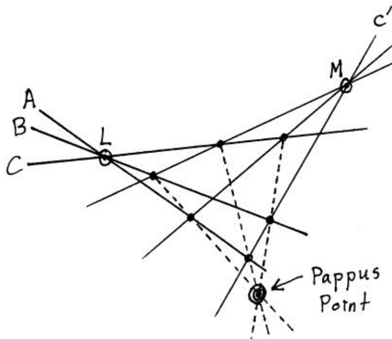
Theorem of Pappus

- Choose two (white) lines
- Choose three points on each line. Label them A, B, C & A', B', C'.
- Draw lines that join these points:
  - AB' and A'B (green lines)
  - BC' and B'C (orange lines)
  - AC' and A'C (blue lines)
- Mark points of intersection (in yellow) of the same-colored lines.
- The three (yellow) points should be collinear. Draw a (pink) line through them.

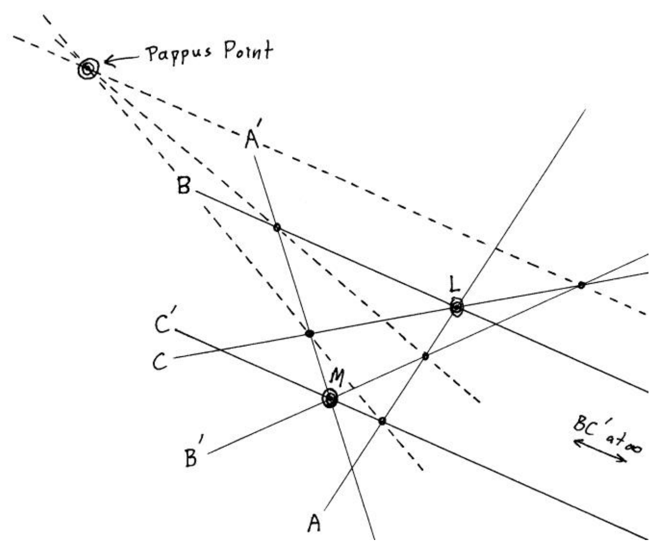
Dual of Pappus

- Choose two (white) points
- Choose three lines through each point. Label them a, b, c & a', b', c'.
- Find points where these lines intersect:
  - ab' and a'b (green points)
  - bc' and b'c (orange points)
  - ac' and a'c (blue points)
- Draw lines (in yellow) that join same-colored points.
- The three (yellow) lines should be concurrent. Mark a (pink) point where they intersect.

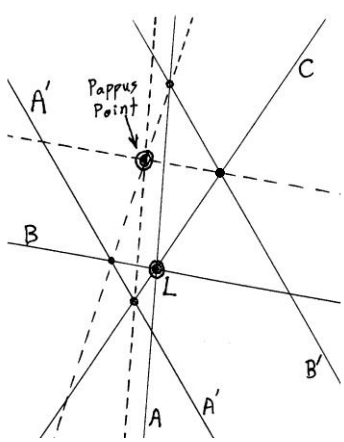
  - Ideas for playing with infinity. What happens if point M is at infinity? Can you even start with both points, L and M, at infinity? (Yes, you can!) What would that look like?
- Hanging Question: There is a circle floating horizontally in front of the blackboard. What are the possible curves that could result by doing a perspectivity of it onto the blackboard?
- Main Lesson Book: Finish up drawings of Pascal, Brianchon, and the dual of Pappus.



Normal drawing of the Dual of Pappus with everything falling conveniently on the page.



Drawing of the dual of Pappus where I have made B parallel to C'. Notice that the line connecting points B'C and BC' passes through B'C and heads in the direction of BC', which means that it must be parallel to lines B and C'. The Pappus point appears nicely on the page.



Playing with infinity. In this dual of Pappus drawing, point M is at infinity (which is also the place where lines A' and B' meet), and line C' is the line at infinity. The Pappus point is on the page!

Day #7

- Review:
  - The daily standard question: "What new and interesting idea can you recall from last week?"
  - Key concepts needing to be reviewed from last week (at least briefly mentioned):
    - Our "postulate" that 2 parallel lines meet at one point at infinity.
    - The line at infinity
    - The three conics: ellipse, parabola, hyperbola
    - Perspectivity (mention that this will soon be built upon)
    - Duality, which is a central principle of PG. In PG can be stated also in its dual form.
    - Regarding two parallel lines: since we can't prove what happens, we simply have to assume something. If we assume "2 parallel lines never meet" then we get Euclidean geometry, and can thereby prove many things (e.g., $\angle$'s in a Δ Σ 180°, or the Pythag Th., etc.). But if we assume something different like "2 parallel lines meet at ∞", then we get a very different geometry.
    - Isabelle's question: "How was it possible that the world of mathematics went from believing Euclid to believing this crazy projective geometry stuff?" (The answer to this will wait!)
- Playing with infinity (with the Dual of Pappus). Do each of the following (if you haven't already):
    - (1) Make the three lines a', b', c' parallel to a, b, c. (2) Have one of the original dots at infinity. (3) Have both of the original dots at infinity. (4) Have one of the original dots at infinity, and have one of the lines (a') through this dot be the line at infinity.
- Statement for the Dual of Pappus:

  Given any three lines (a, b, c) *on one point and another three lines* (a', b', c') *on another point, the three corresponding pairs of points* (ab' & a'b, ac' & a'c, bc' & b'c) *can be joined by concurrent lines.*
- Discussion.
  - Ask: "What if someone now were to ask you 'What is projective geometry?'
  - Any given triangle can be transformed through a projectivity to take the shape of any other triangle. Every triangle is identical to every other triangle. In PG we can say that
    Every triangle is "congruent" to every other triangle.
    - Show my High Mowing photo (of a triangle on the window and its shadow) to show perspectivity.
- Hanging question (from previous day): Perspectivity of a Circle
  - Show how a circle (horizontally in space in front of the chalkboard) can be perspected onto the board as an ellipse, parabola, or hyperbola depending on the location of the point of perspectivity. This leads to this thought: Whether a conic is an ellipse, a parabola, or a hyperbola depends on how many times it touches the line at infinity.
- Projectivity. Brief intro only. *A projectivity is a series of two or more perspectivities*.
- Theorem of Desargues: **(This can be skipped if time is short in this block.)**
  - Show two random triangles and that connecting corresponding points doesn't create concurrent lines. Ask: how can we create two triangles that are "perspective to a point"? By working backwards!
  - Create this ML Book page:
    - *Perspective to a Point*. Include a drawing which shows 2 triangles which are perspective to a point, with the statement: "Corresponding points form concurrent lines." Do the drawing by working backwards. Start with the point of perspectivity, then the three connector lines. Then chose the points of the triangles on these three lines.
    - *Perspective to a Line*. Include a drawing which shows 2 triangles which are perspective to a line, with the statement: "Corresponding lines meet at collinear points." Do the drawing by working backwards. Start with the line of perspectivity, then choose three points on that line. Through each of these three points draw a red line and a green line. You then have two triangles that are perspective to a line.
- Hanging Questions: How can a line be parallel to two lines which aren't parallel to each other?
- (Only for those needing a challenge – see website) *Projective Geometry Puzzle #3 – Pappus in 3-D.*
- Main Lesson Book: Two drawings of Desargues.

Day #8

- Review:
  - The daily standard question: “What new and interesting idea did you walk away with yesterday?”
  - Answer yesterday's hanging question(s).
    - Parallel lines can be defined as two co-planar lines that meet on the line at infinity. Since every line crosses this "special" line somewhere, we can say that all lines meet the line at infinity at a point on the line at infinity, which, by definition, means that the two lines are parallel.
    - Euclid’s theorem I-30 states: “Two lines that are parallel to the same line are parallel to each other.” This theorem is proved by using the parallel postulate. Thus, this theorem is not valid in PG.
    - We have trouble believing that every line is parallel to the line at infinity because if we look at a line that is running from our feet out toward the horizon (the line at infinity), then these two lines look perpendicular. However, we can turn ourselves 90°, take a few steps back, and “see” that the line on the ground in front of us appears parallel to the horizon (the line at infinity). In this way, **any line can be made to look parallel or perpendicular to the line at infinity**.

Drawings from yesterday: Have drawings on the board before class that review what was done yesterday, especially including drawing that involve playing with infinity.

- Are you feeling tortured? Are you struggling with some of these ideas? Wolfgang Farkas Bolyai's advice in a letter (ca. 1830) to his son, Janos Bolyai, who was in the process of discovering non-Euclidean geometry:

  > *"Fear it no less than the sensual passions because it, too, may take up all your time and deprive you of your health, peace of mind, and happiness. For God's sake, please give it up!"*
- Theorem of Desargues: (cont.) (Again, this can be left out if time is short.)
  - Lead the students toward the realization of the Theorem of Desargues.
  - Have a few drawings of Desargues on the board before class, including one that looks like a tetrahedron that has been sectioned twice to show that a drawing can be seen as 2-D or 3-D. With each drawing just have the two triangles drawn before class. Complete the drawing during class.
  - Give statement: *If two triangles are perspective to a point, then they are also perspective to a line.*
  - Include cases when one side is parallel, and when vertices are on different sides.
- (If time allows) Show a drawing (have it on the board before class) that has a series of four perspectivities. Mention that it can be reduced to two perspectivities.
- In Groups: Have the students do the dual of the Desargues drawing.
- A Special Projectivity: if you are running a bit short on time, the students don’t need to do this – just show the drawing on the board.
  - Consider a projectivity of the points 1 through 10 on line x and points 1 through 10 on line y constructed as a series of two perspectivities with perspective points P and Q and intermediate line ℓ. Arrange it so that the line PQ is concurrent with lines x and y (i.e., if the point of intersection of lines x and y is labeled R, then P, Q and R are collinear). Draw all of the connector lines (i.e., lines that connect corresponding points on x and y).
  - What do you observe? Answer:
    - The connector lines are all concurrent.
    - This means that the projectivity has been reduced to a perspectivity.
  - Statement for the drawing: “With a one-dimensional projectivity, if P, Q, and xy all collinear, then the connector lines are concurrent. The whole projectivity can therefore be reduced to a perspectivity.”
- Hanging Questions:
  - What happens with Th. of Desargues when all corresponding sides are parallel?
  - Does the Th. of Desargues work for a quadrangle (or a polygon with more than three vertices)? (Answer: no.)
  - (The Bridge to constructing a line-wise conic) With the drawing *A Special Projectivity,* what happens to the connector lines if something moves so that PQ doesn't pass through xy?
- Main Lesson Book: Finish the drawing: *A Special Projectivity*
  (Extra credit possibilities: Page on projectivity. Dual drawing of a projectivity.)

Day #9

- "Out of Nothing, I have created a strange new universe." -- Janos Bolyai regarding non-Euclidean geom.
- Review:
  - Go over yesterday's hanging question(s).
  - Whether a conic is an ellipse, a parabola, or a hyperbola depends on how many times it touches the line at infinity. (This shows again that the special status of the line at infinity is fairly arbitrary.)
- Characterizations:
  - Just like there are no squares, rectangles, etc. in PG, only quadrilaterals and quadrangles, so too, in PG, there are no circles, ellipses, parabolas (except to our Euclidean eyes) – there are only conics.
  - In PG there is no such thing as a circle. There is only conic. And any conic can be perspected or projected (from one plane to another) to look like any other conic.
- Experimenting day! Most of this day is spent experimenting with line-wise conics.
- Prioritizing Drawings. With this topic, there are five drawings:
  - *Special Projectivity*. This was done yesterday, and is an optional drawing for the students.
  - *Line-wise Conic from Scratch*. This is the project for today, where the initial lines (x,y,ℓ) and points (P,Q) are all chosen at random. If there is a lot of time, students can do several of these drawings. If you are short of time, students can just try one of these – or this drawing could be skipped.
  - *Line-wise Conic using a Template.* This is the most important drawing because the students can see how the curve transforms.
  - *Line-wise Parabola*. Great thing for everyone to do, if time allows. Just start with line Y at infinity!
  - *Point-wise Conic*. Just do the dual of a line-wise conic. Great for those needing an extra challenge.
- Line-wise Conic (from scratch).
  - Show that with *A Special Projectivity* if we had somehow instead mark all of the (infinitely many) points on line x, and if all of the (infinitely many) connector lines were blue, then the whole page would have been painted solid blue.
  - Introduce this by first talking about the *Special Projectivity* drawing as covered with a blue "fog". Once we move the position of one of the original points (P,Q) or lines (x,y), then a bit of the fog will be cleared so that a form arises.
  - The process for constructing a line-wise conic is identical to the process for *A Special Projectivity* (see previous day), except that we intentionally set it up so that the line through P and Q is not concurrent with lines x and y.
- **Tips for the Line-Wise Conic drawings:**
  - Expect that it won't work out on the first attempt and you will need to start over again at least once.
  - Color code: x, y, P, Q and ℓ are orange on the board and black ink on paper. Construction lines are white on the board and lightly in lead on paper. The final connector lines are all blue.
  - It can be frustrating to start with many points on x, perspecting to ℓ and then y (through P and Q), and then finding out, after all that work, that the results are off the page. So here is what can be done…
    - Start with just three points (1,2,3) on line x, and locate those three points on the intermediate line ℓ.
    - With light lead pencil, draw line y and point Q. From line ℓ and through point Q, locate the three points on line y. Lightly with lead pencil, draw the three connector lines (lines that connect corresponding points on x and y). The goal is to get two connector lines to intersect on the page.
    - If no connector lines intersect on the page, then erase everything that has been done in lead pencil. Now think about where Q and y could be placed such that the entire previous step could be redone and it would result in the connector lines intersecting on the page.
    - Once you are confident that the drawing will fall nicely on the page, then you should try to start with about 10 points on line x. Don't make these points evenly spread out, but instead have the lines through point P with equal angles between them.
  - Note: there are two ways to draw a line (1) given two points; (2) given one point and a direction.
  - Dealing with points off the page. Sometimes a point (call it A) on line x gets perspected (through point P) onto line ℓ such that it falls off the page. We can then temporarily tape a page on so that we can accurately locate the point on line ℓ. Often this point is then perspected back onto the page onto line y.
  - Show how to draw parallel lines using a ruler and a (plastic) right triangle.
  - The students should be required to **deal with points at infinity on x, y, and ℓ**. These 3 points are very useful to complete the holes in the drawing.
  - Mention that each of these lines are on the curve: x, y, the line connecting P and Q, the line connecting ℓ x with Q, and the line connecting ℓ y with P.
  - Line-wise conic Statement: "Normally, the connector lines of a projectivity form a line-wise conic."
- (Only for those needing a challenge – see website) *Projective Geometry Puzzle #2 – Fundamental Thereom*.

- Hanging Questions:
    - How can I construct a drawing that results in a line-wise parabola?
    - The distance to the sun. How is it possible to conclude that the sun is much further away (and larger) than the moon by just using observation and thinking? (Perhaps save this until later in the block, or later in the year, when there is a half moon, and the moon and sun are both visible in the sky.)
      Answer: We can imagine that there is a vector (arrow) which has its tail at the center of the half of the moon that is lit. This vector always points directly at the sun. However, when there is a half-moon it doesn't look like it points to the sun. It makes sense if you point your arm to the sun. Your arm appears to be parallel to the moon vector, which only makes sense if you think of the sun close to infinity.

Day #10

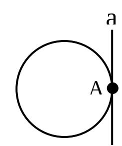
- Review
    - Reiterate that duality is a fundamental principle of projective geometry.
    - Review by showing a drawing of a line-wise conic on the board. Emphasize how the connector lines (which form the conic) act as a huge piece of chalk, shading in all of the plane except where the conic is. Show this by tracing the movement of the connector lines with a yardstick, where the yardstick moves from connector line #1, to line #2, to line #3, etc.
- Discussion. Are we doing real math? Does infinity even exist? Does anything in math exist?
- **Line-wise Conic using a Template**:
  This is perhaps the most important drawing because it will allow us to see the conic in movement.
  Instructions:
    - Make copies of the file "linewise conic template.pdf" and give one copy to each student.
    - Everyone will have the same lines X and ℓ, and the same points P and Q. The only difference will be the location of Line Y.
    - The central idea is that Line Y (which is horizontal) is moving upward on the page. How does this affect the shape of the curve?
    - It is important to note that there are three places where the curve degenerates into a perspectivity:
        - When points P, Q, and the intersection of lines X and Y are all collinear. This happens when Line Y is 12.2 cm above the bottom of the page. (This was our *Special Projectivity*.)
        - When lines X, Y, and ℓ are all concurrent. This happens when Line Y is 14.1 cm above the bottom of the page.
        - When point Q is on Line Y. Line Y is 18.7 cm above the bottom of the page.
    - Assign each student a distance (between 1 and 28 cm) which determines how far line Y (which is horizontal) is above the bottom of the page. Be sure not to assign a distance that is closer than 0.6 cm to the degenerate cases (12.2, 14.1, or 18.7, as explained above).
      For example, these are the values I assign for a class of 18 students:
      4.2, 5.1, 6.1, 7.1, 8.1, 9.1, 10.1, 11.0, (12.2 is degenerate), 12.8, 13.1, 13.4,
      (14.1 is degenerate), 15.5, 16.3, 17.2, (18.7 is degenerate), 19.8, 21.6, 23.3, 24.9
    - It is best if everyone in the class colors in their curve with the same colored pencil.
- Line-wise Parabola. How can we produce a parabola with the line-wise conic drawing?
  Ans: If we move line y, line x, or both of the points P and Q to infinity then it must be a parabola.
  Have the students do this drawing.
- (For those needing a challenge) Do a point-wise conic by dualizing the line-wise conic. Directions (for those who need it):

Line-wise Conic	**Point-wise Conic**
1) Find 20 points on line x.	1) Draw 20 lines on point X.
2) From each one of these 20 points on x, draw a line passing through P.	2) With each one of these 20 lines on X, find where it intersects with line p.
3) With each one of these 20 lines on P, find where it intersects with line ℓ.	3) From each one of these 20 points on p, draw a line passing through L.
4) From each one of these 20 points on ℓ, draw a line passing through Q.	4) With each one of these 20 lines on L, find where it intersects with line q.
5) With each one of these 20 lines on Q, find where it intersects with line y.	5) From each one of these 20 points on q, draw a line passing through Y.
6) Draw the 20 connecting lines from each point on x to its corresponding point on y.	6) Find the 20 connecting points by finding the points of intersection of each line on X and its corresponding line on Y.

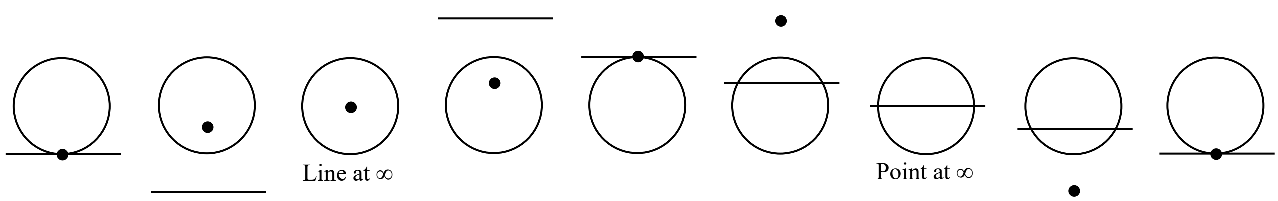
- Challenge: (1) Construct a pointwise conic; (2) Construct a parabola (linewise or pointwise)

Day #11

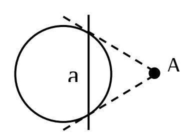
- Review
  - What is left of geometry if you eliminate everything having to do with measurement? Answer: projective geometry! In PG we never deal with measurement. We are answering the question: "What laws in geometry are still true without measurement?"
  - Show the students' *line-wise conic drawings using a template* that transforms the conic as line y moves.
  - Intro to Polarity.
    - Generally speaking, we can say that polarity is "specific duality".
    - Start by showing the result of using a regular hexagon with the Theorem of Pascal (where the Pascal line is at the line at infinity) and the Theorem of Brianchon (where the Brianchon point is at the center of the circle) Speak of this point at the center as the "infinitely intensive or infinitely inward point" – it is a different sort of infinity.
    - Mention that our major goal is to be able to take a curve and "turn it inside-out"
    - Refer to the conic that you are taking all polarities with respect to as the "polarity conic". This polarity conic serves as a kind of mirror. Mention that we will use a circle as the polarity conic, but it could be any conic.
    - Start with these fundamental ideas:



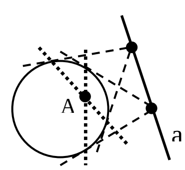
      - The polar of the "infinitely distant line" (the line at infinity) is the "infinitely inward point" (the center of the circle).
      - The polar to a point (A) on the polarity conic is the line (a) tangent to the polarity conic at that same point, as shown here.
      - *Important imagination*. The below 9-frame "video" shows what happens as line moves.



    - Four exercises – each one to be discovered by the students:



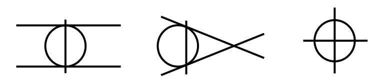
      1. The polar of a point outside the circle is a line "inside" (i.e., crosses) the circle. We find that line by drawing the two tangent lines to the circle and connecting the two points of tangency.
      2. The polar of a line that crosses the circle. (Same drawing as for #1.)
      3. The polar of a line outside the circle.
      4. The polar of a point inside the circle. (Same drawing as #3.)



      Note that #3 and #4 are the most difficult.
    - Important note: Emphasize that you shouldn't just locate the point of tangency "by eye". Show how to find the point of tangency (on a circle) of a line that passes through a given point outside the circle by using a drawing triangle. (The point of the right angle goes on the circle; one leg of the triangle passes through the point, and the other leg passes through the circle's center.) Of course, we should be aware that we are cheating a bit, because in PG there are no right angles, and, for that matter, no circles. We allow for this because we know (how?) it is possible to construct a tangent to a conic by using only a straightedge. (This appears later as "The Tangent Puzzle".)
  - Puzzle Problem: How can I construct a self-polar triangle? Likely, someone will find a specific case. If so, tell them to now find a general case – with no points or lines on the conic or at infinity.
- Hanging Question(s):
  - How can a self-polar triangle be constructed?
- Main Lesson Book:
  - A page on *The Basics of Polarity,* which includes an explanation of polarity.

Day #12

- Review
  - Discussion. Ask: "If someone now were to ask you 'What is projective geometry?', how would you answer?"
  - Central Question: what has been in common with all of our PG theorems thus far? This should lead to one important idea: In projective geometry there is no concept of measurement. Every PG theorem (and drawing) has nothing to do with measurement.
  - Show the animated transformation of a line-wise conic on the computer (this is a Geometer's Sketchpad graphic animation which can be downloaded from my website).
- Polarity
  - Show a variety of triangles and their polar triangle. Have these on the board before class. Include cases where the triangles overlap, and where two of the lines of the triangle are (or end up) parallel.
  - Go over hanging question: Constructing a self-polar triangle
    - Show the various "Special possibilities", including:



    - General procedure for a finding a self-polar triangle: (1) choose any point, A, and find its polar line, a; (2) Choose any point, B, on line a, and find the polar line, b, to point B; (3) point C is the point common to lines a and b; line c is the line common to points A and B.
- Drawing of the Day: Polarity of a Trefoil.
  - See "Notes on the Polarity of a Trefoil" at the end of these lesson plans.
  - Everyone should do the same drawing (as shown in the notes at the end of these lesson plans), where the trefoil is inscribed inside the polarity conic (circle). The symmetry simplifies matters considerably.
  - To some degree, because this is the first polarity drawing, this is a planned disaster.
  - The intention may not be to produce a polished drawing, but rather to learn from the experience and then apply what we have learned to the next more difficult polarity drawing.
- **Important Notes to consider if you wish to do Polarity of a Curve:**
  - Much of the rest of the next three days can be spent doing the polarity of various curves.
  - Everyone should at least do the polarity of a trefoil (see above). Ideally, it is nice if everyone can also do the polarity of another, more complicated curve, such as a limaçon, cardioid, or lemniscate. However, it may be too much for everyone to do this. Maybe it is enough for some people to just do the trefoil.
  - The highest priority should be to wrap up the course and ensure that everything ends well. You may elect to spend less time on the polarity of curves and more time instead on other topics (Chasles' Theorem, various puzzles, etc.).
  - "Tips for Constructing the Polarity of a Curve" and "Polarity of a Trefoil" are both copied at the end of these lesson plans. This is all taken from my High School Source Book.
- Hanging Questions:
  - How would the PG troll (who only knows about projective geometry) answer the question: "Do parallel lines meet at infinity?"

Day #13

- Go over yesterday's hanging question(s).
- **The "Final Project": The Polarity of a Curve** (This normally takes three main lesson periods)
  Here is the sequence of events for making this work:
  - Have the whole class do the polarity of a trefoil – which was done yesterday.
  - Go over details of "Tips for Constructing the Polarity of a Curve" (see end of these lesson plans).
  - It is best to carefully plan which drawing to give to each student. I like to do sequences with particular curves so that once the drawings are complete we can see the curve in movement. If you do this, it is best to use the polarity templates from my website since I have ensured that everything falls on the page nicely.
  - I put these drawings on the wall in the Math Room for one year. It is one of the rare times that I display student work on the walls.

- <u>Step-by-Step Process for Completing a Drawing for the Polarity of a Curve</u>. Write the following steps on the board:
  1. Finding where the polar curve passes through infinity:
     a. *Lightly* draw all tangents lines to the curve that pass through the center of the circle.
     b. The polar of each of these tangent lines is a point at infinity in the direction that is perpendicular to the tangent line. In each case, show an arrow (with a long tail) at the edge of the page (lightly in pencil) that indicates the direction that the polar curve goes to infinity. Show it also in the opposite direction on the other side of the page. We will number these arrows later.
     c. With each of these tangent lines (that pass through the center of the circle) mark the points of tangency on the original curve.
     d. Erase all tangent lines.
  2. Mark around 20 points on the curve (some of which come from the above special tangent lines). Carefully think about where the points need to be closer together. Label them in order.
  3. Label the arrows (from step #1b) with its proper number (which should corresponds to the point of the curve that had a tangent line passing through the center of the circle).
  4. Finding the Polar Points:
     a. Choose one point on the curve and *carefully* draw (lightly in pencil) a tangent line through that point.
     b. Find the polar point, and mark it in ink. Label it with the correct number. Ask yourself if it makes sense!
     c. Erase the tangent line, all constructions lines, and any tick marks.
     d. Repeat the process for all points on the original curve until all corresponding points of the polar curve have been located. It may become apparent that more points are needed.
  5. Lightly in pencil, connect the points of the polar curve. Pay special attention to the direction(s) that the curve goes to infinity (given by the arrows in step #1). Ask the teacher to check your work.
  6. Color the curve nicely with the required colors.
  7. *Very neatly*, add the title (2.5cm height) and your name (1.5cm height). Make sure the page is in the proper orientation – up is up!

- <u>The PG Troll skit</u> where a confused PS student speaks with the PG Troll.
  - Central to the skit is that the student asks the PG troll (who only knows about projective geometry): "Do parallel lines really meet at infinity?"
  - At some point the student is trying to explain to the troll what parallel lines are. The troll is then shown two parallel lines on the floor.
  - The troll simply asks: "What's so special about them."
  - The student says: "Well, I want to know if there is still a point of intersection."
  - Troll: "Of course there is. There always is."
  - Student: "But I can't see it!"
  - Troll: "Well then, just do a perspectivity that brings the lines onto the board."
  - The conversation continues and we learn that the PG troll has no idea what parallel lines are, or what infinity is, which leads to the realization that…
  - PG doesn't deal with measurement. Implications: no squares, no right angles, no parallel lines; no such thing as infinity. The infinitely distant point of a line does exist, but in PG it isn't considered anything special because it can easily be perspected onto another line that can be "brought onto the page."
  - In true PG…
    - Any given triangle can be transformed through a projectivity (perspectivity?) to take the shape of any other triangle. Every triangle is identical to every other triangle. In PG we can say that <u>**Every triangle is "congruent" to every other triangle**</u>.
    - Be careful not to say, "there is only one triangle" – because there are, of course, infinitely many triangles and infinitely many conics in the plane – but they are all "congruent" to each other.
    - <u>**Every conic is "congruent" to every other conic**</u>.
    - <u>**Every pair of (coplanar) lines is "congruent" to every other pair of (coplanar) lines**</u>.
    - <u>**Any two coplanar lines can be made to look as we pleased. There are no parallel lines!**</u>
    - However a form happens to look at one given instant, is a "Euclidean" picture of that form.

- <u>Hanging Question(s):</u>
  - Now we have heard (1) That parallel lines never meet; (2) that parallel lines meet at infinity; (3) that there is no such as parallel lines. *So what is really the truth about parallel lines?*

Day #14 Review, finish drawings, and do extra drawings.

- The Big Question: After having heard the message from the PG troll, what is the truth regarding parallel lines?
  - Is it true that two parallel lines intersect at infinity? Yes, there is some truth to it, and there is some truth to Euclid's 5th postulate. We can't definitively say what the truth is. PG cleverly avoids the issue.
    - Two key postulates of PG:
      - Any two points have a common line.
      - (In a plane) any two lines have a common point.
  - In PG there is no such thing as: parallel, or infinity.
  - In PG the line at infinity exists (which doesn't in EG), but it isn't given any special status, since it can just be perspected into the "middle of the page" of another plane. (This is what an artist does with the horizon line.)
  - After Euclid, there was essentially nothing completely new with geometry for almost 2000 years.
  - Then Desargues and Descartes each offer a new geometry to the mathematical world. Descartes' new geometry adopted, and it had a profound influence on the development of Western civilization. Desargues' geometry was ignored. Perhaps he was too far ahead of his time. How would things have evolved if projective geometry had been adopted instead of analytical geometry?
  - Arthur Zajonc comment: "Sequential, logical, certain thinking is from the previous epoch. This epoch demands a flexible, relative, paradoxical thinking." This is also true of modern physics.
- Continued work on polarity drawings and any other drawings that need to be completed.
- Final Hanging Question (until next year with the Philosophy of Math):
  What replaces Euclid as the foundation of mathematical truth?

Day #15

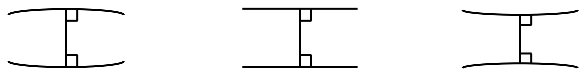
- Continued work on polarity drawings and any other drawings that need to be completed.
- Ask the students: Why do you think Rudolf Steiner thought it was a good idea to teach projective geometry to high school students?

Material to consider if time allows

- Chasles' Theorem: "A triangle and its polar triangle are always in perspective." (from Coxeter, p64).
  - Have this come from the students. Ask: what is special about any triangle and its polar? Perhaps give them the hint to drawing lines connecting corresponding points of the two triangles.
- The Inscribed Quadrangle Theorem: "With any quadrangle inscribed in a conic, its diagonal triangle is self-polar." (from Coxeter, p75).
  Do the following exercise so that the students can discover this for themselves:
  (1) Choose four points on circle (conic).
  (2) Draw all 6 connecting lines.
  (3) Locate the three points of the diagonal triangle.
  (4) Find the polar (w.r.t. the conic) of this diagonal triangle.
  (5) You should then notice what the theorem is.
  (6) (optional, but nice!) Do the dual drawing of the above.
- Mention that the dual of the Inscribed Quadrangle Theorem is the Circumscribed Quadrilateral Theorem.
- *Projective Geometry Puzzle #4*. All of these puzzles involve the Inscribed Quadrangle Theorem.
  Part 1: Find a method for constructing the polar line to any given point with respect to the polarity conic, without drawing tangent lines, or using a drawing triangle. This method should work whether the given point starts inside the polarity conic, or outside, and should work for a non-circular polarity conic. Also, find a method for constructing the polar of a line.
  Part 2: Find a method to construct a tangent line to a conic through a given point outside that conic.
  Part 3: Find a method to construct a tangent line to a conic through a given point on that conic.

- Polarity of conics. Take the polar of both line-wise conics and point-wise conics. This will take a good amount of time.
  - How can we take the polar of a conic and produce a parabola? **Answer:** We just need to make sure that one (and only one) tangent line of the original conic passes through the center of the polarity circle.
- *Inside and outside a conic.* Even though there is no such thing as the "inside" of a triangle, there is an "inside" of a conic.
  - The conic divides the plane into two regions: Inside and outside.
  - Any two lines that are tangent to the conic will always intersect at a point that is outside the conic.
  - The dual of the above statement is: Any two points on the conic are joined by a line that is always inside the conic.
  - A line outside the conic has no points on the conic.
  - The dual of the above statement is: a point inside the conic has no lines that are tangent to the conic.
- When doing a polarity of a conic, how do I know if the outcome will be an ellipse, hyperbola, or parabola? Answer: It depends only on where I place the center of the polarity circle. If this center is inside the conic (that we are taking the polar of), then no tangent lines of the original curve pass through the center, which then means that none of the points of the polar curve will pass through infinity. If two tangent lines of the original curve pass through the center of the polarity circle, then the resulting curve has to be a hyperbola, because it will have two points at infinity. The only way to get a parabola is if the center of the polarity circle lies exactly on the original conic (that we are taking the polar of).
- *Perspectives of a conic*. John is standing at a certain location on a plane looking at a conic section. He is standing at the vertex of the conic if it is a parabola, and at the "center" of it if it is an ellipse or hyperbola. Beth is standing on the horizon (according to the perspective of John) on the axis of symmetry of the conic. The conics are somehow large and bright enough that both of them can see the entire conic clearly. What does each of them see? Consider the case of the ellipse, parabola and hyperbola.
  - *Perspectives of a conic*. Answer:
    - If it is a parabola, then they both see it as an ellipse. John and Beth on standing on opposite ends of the parabola (which looks like an ellipse).
    - If it is an ellipse (and John is standing at the center), then I'm not sure what Beth sees. Perhaps it is a hyperbola to her, where she is standing at its center (one branch is in front of her, and one is behind her). Or maybe it looks like a ellipse to Beth whether she looks forward or backward.
    - If it is a hyperbola (from John's perspective), then it John would see Beth standing at the center of each branch of the hyperbola. (Since Beth is "on the horizon", John sees her on two places 180° apart.) From Beth's perspective, it is an ellipse.
- History of Projective Geometry.
  - **17th Century** – The first steps
    - ca 1610: Kepler (propeller)
    - 1637: Descartes publishes *Discourse on the Method*.r
    - 1639: Desargues publishes *Brouillon Project*.
  - **18th Century** – Sleep.
    - PG goes to sleep for nearly 200 years, at a time when the rest of the world of mathematics takes off.
  - **19th Century** – Conscious development of projective geometry
    - 1826 Gergonne discovers the law of duality.
    - ca. 1825 Gauss discovers some of the laws of PG and non-Euclidean geometry, but doesn't publish he ideas for fear of ridicule.
    - 1829 Nikolai Lobachevsky (Russian) publishes first book on non-Euclidean (hyperbolic) geometry.
    - 1831 Janos Bolyai writes an appendix to his father's book on non-Euclidean (hyperbolic) geometry.
    - There is an explosion of interest in synthetic ("pure") geometry involving many mathematicians, including Chasles, Brianchon, Poncelet (a general in Napoleon's army), Jacob Stenier.
    - Then the mathematical community became divided between those who felt that geometry had become too analytical, and were thrilled with the new revival in "pure" (synthetic) geometry, and those who felt that much of the new geometry was somewhat reckless.
    - There became a renewed interest in proving Euclid's fifth postulate based upon the first four. This had most notably been attempted by Omar Khayyam (ca. 1080) and Giralamo Saccheri (ca. 1700). Only after significant developments had been made with the new geometries (each one assuming an alternative to the fifth postulate, and yielding different, but equally valid geometries), in 1868 Eugenio Beltrami proved that it is impossible to prove the fifth postulate based upon the first four.
    - This led to the terrible realization that the 5th postulate "isn't necessarily true" and the "Foundational Crisis" (teaser for next year's course on *The Philosophy of Mathematics*): If Euclid is no longer the unshakable foundation for the world of mathematics, then what will replace it?

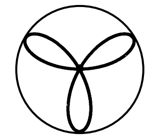
- <u>Non-Euclidean vs. projective geometry</u>
  - **<u>Summary of non-Euclidean geometry</u>**.
    - The 5th postulate can be reframed to *Playfair's Axiom*, which asks the question:
      "Given a line and a point not on that line, how many lines can be drawn (on the same plane) through the point that do not intersect with the given line?"
      - If the answer is "one line is possible", then we get *Euclidean geometry*.
      - If the answer is "no lines are possible", then we get *elliptic geometry*.
      - If the answer is "infinitely many lines are possible", then we get *hyperbolic geometry*.
    - Visually, these three different geometries can be roughly represented as shown here:

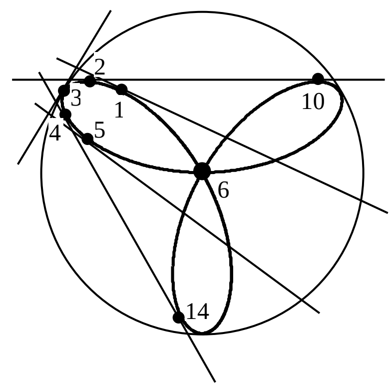
      Elliptic **Euclidean** **Hyperbolic**
    - Elliptic and Hyperbolic geometry are essentially making different assumptions than the 5th postulate, which results in different mathematical "truths".
    - With PG, there is no assumption made about the 5th postulate – Playfair's question is not a valid question. Therefore all theorems of PG should also be true (perhaps with some minor tweaking) in each of the other three geometries. This is why Arthur Cayley (1859) said: **"PG is all geometry."**
    - It is interesting to note that Einstein said hyperbolic (non-Euclidean) geometry is a better representation of reality in astro-physics (super-large spaces).
- <u>How can I find the center of any conic</u>?
  <u>Answer</u>: Given that we can now find tangent lines to a conic (from any point not on that conic), we can find the center of the conic by taking the polar of the line at infinity. We start by taking two points (P and Q) on the line at infinity. From each of these two points, we follow the process given on the "tangent puzzle". First, starting from point P, we draw two secant lines (which intersect the curve in two places each), which gives us a quadrangle on the conic. Then we construct the diagonal triangle of the quadrangle, which gives us the polar line of point P. After finding the polar line of point Q in the same manner, we know that the conic's center (which is the polar of the line at infinity – the line connecting P and Q) is the point of intersection where these two polar lines meet.
  If we complete drawings following this process, we see that the ellipse has its center where we'd expect, and the hyperbola also has its center where we'd expect (at the intersection of its asymptotes). We also notice that the center of the hyperbola is outside the hyperbola. The center of the parabola is on the line of infinity at the point of tangency. Because the parabola is the instant between an ellipse (with a center inside) and a hyperbola (with a center outside), it makes sense that the center of the parabola is actually on the curve.
- <u>The Fundamental Theorem of Projective Geometry</u>.
  "*A projectivity is fully determined by three points on one line, and three corresponding points on another line.*" In other words, once you know 3 pairs of corresponding points, any other point on either line has a fixed place where its corresponding point on the other line must occur.
  - I am not convinced that it is more important, or more pedagogically valuable than any of the other topics that I have managed to include. It seems to not have the "bang" for the 11th graders that it has when I teach it to an adult course.
- Show how a series of four perspectivities can be reduced to two perspectivities. Erase old intermediate lines and points, and then use the Pappus line for line ℓ and then use A' and A for points P and Q.
- The axis of projectivity (drawings C-10)
- Projectivity of points onto the same line (i.e. co-basal projectivities, see Edwards, p46).
- Harmonic conjugates and involution.
  - Compare finding the harmonic conjugate of a point by constructing a quadrangle with Desargues's method, which was measuring – he stated that pairs of harmonic conjugates were such that the product of the distances of the two points to the center of the involution (halfway between the two fixed points) is constant. I.e. if O is the center (the midpoint of XY), then
    OA•OA' = OB•OB' = OC•OC', etc., where A, A' and B,B' and C,C', etc. are harmonic conjugates.
  - Imaginative exercise: Given an involution with two fixed points, X and Y, and a point, A, that is in movement, and its harmonic conjugate, B. Picture the movement of A and B together.
- Perspect a hyperbola and its two asymptotes from the floor onto the board. This shows that a hyperbola is tangent to its asymptotes, but surprisingly <u>doesn't</u> cross them.
  - (see Edwards, 2nd Edition, p131-3 Fig 11.1-11.6)

Notes on the Polarity of a Trefoil (Taken from my High School Source Book)

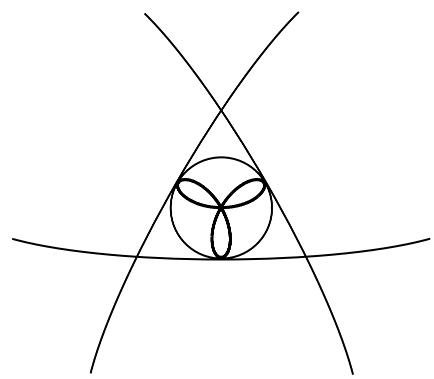
- This is a great first drawing to do for the polarity of a curve. It is relatively easy to construct (because of its symmetry and because all of the tangent lines cross the polarity conic), but it has great thinking behind it.
- Place the polarity conic so that it is tangent to all three pedals, as shown on the right. Make the polarity conic's diameter no larger than about 30% of the width of the page. (You can copy this from my polarity templates in the following pages, or download it from our website.)



- Most of the "locations" on the curve have tangent lines close to the center of the polarity conic, which result in polar points landing off the page. Therefore, we will only work with locations that are close to the tips of the pedals.
- Working with the natural symmetry of the original curve, we will take the polar of five tangent lines on each pedal, thereby producing a total of 18 points on the resulting polar curve. Points 6, 12, and 18 are all (on top of each other) at the center of the circle.



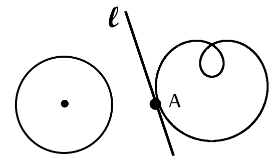
- Note that points 2 and 10 share a common tangent line. So do points 4 and 14, and 8 and 16. This means that the polar of each one of these tangent lines (with two points on it) will be a point with two tangent lines through it – i.e., the polar curve will pass through these points twice.
- The tips of each pedal (points 3, 9, 15) will be self polar (i.e., the original curve and the resulting polar curve will touch one another at these locations).
- The most interesting question is: what happens to the polar curve when the original curve is at the center of the polarity conic (i.e., locations 6, 12, 18)? For starters, we know that because the tangent lines here pass through the center of the polarity conic, then the polar points must be at infinity in a direction that is perpendicular to the tangent lines. But if we observe closely what happens if we travel along the original curve through the center point, we will notice that the tangent line touches the center point, but does not pass through it. This must mean that the polar curve has three places that touch the line at infinity, but doesn't pass through infinity – i.e., it is tangent to the line at infinity!

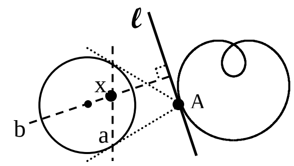


- The final drawing looks like what is shown here on the right.

Tips for Constructing the Polarity of a Curve (Taken from my High School Source Book)

- Instead of thinking of the curve as a collection of points, or, alternatively, as a collection of (tangent) lines, we should think of it as a collection of "locations", where each location on the curve has both a point and a tangent line through that point.
- For most curves, choosing between 25 and 30 points on the original curve should be sufficient. These points should not be equally spread out; the points should be closest together wherever the tangent lines are turning the fastest, and wherever there is a cusp or a point of inflexion. The points should be labeled in numerical order as you travel around the curve. Through each point, tangent lines must be drawn as accurately as possible. Be sure to include tangent lines that pass through the center of the polarity conic, as well as tangent lines that are tangent to both the curve and the polarity conic.
- If we are taking the polar of a tangent line (ℓ) that falls outside the polarity conic, then we need to take the polar of any two points on the line. It is best to have the two points be the point of tangency (A), and the tangent line's infinitely distant point (B). The process is conceptually challenging, but easy to execute. First we take the polar of point A, which is line a (which must cross through the polarity conic). Now we will take the polar of point B (the infinitely distant point on line ℓ). The key is to realize that the polar of a point at infinity is a line that runs through the center of the polarity conic and is *perpendicular to the direction of the point.* Therefore, to get line b, we just draw a perpendicular line to line ℓ, such that it passes through the center of the circle. Now we have found our goal: the polar point to line ℓ; it is point x – the point of intersection of lines a and b. By using a drawing triangle, this can be done very quickly.

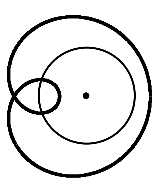




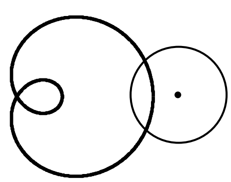
- If we only took the polars of the points of the original curve, then our result would be a collection of tangent lines, and we likely wouldn't be able to tell what the curve really looked like (unless it was a conic). Two very helpful tips: (1) <u>It will simplify matters greatly to only take the polars of the original curve's tangent lines</u>, which will give us the desired points on the polar curve; (2) Draw one tangent line at a time on the original curve, find the polar point, then erase all construction lines.
- Pay special attention to where the original curve's tangent lines pass through the center of the polarity conic. This tells where (the points of) the polar curve must pass through infinity. *If the original curve's tangent line passes through the center of the polarity conic, then the polar curve must pass through infinity in a direction that is perpendicular to that tangent line.*
- By just looking at the original curve and the polarity conic, it is impossible to tell where the resulting curve will fall on the page. Therefore it is best to use my polarity templates (on the following pages or on my website) in order to ensure that the results fall nicely on the page.

Key questions to think about:

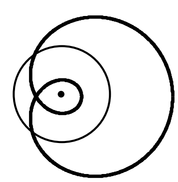
- By looking at the original curve and the polarity conic, how can we tell how many times the resulting curve will pass through infinity? Answer: It depends on the number of times the original curve has tangent lines passing through the center of the polarity conic. To show this clearly, draw three limaçons on the board, as shown below.



The polar curve passes through infinity twice.



The polar curve passes through infinity four times.



The polar curve doesn't pass through infinity at all.

- If the original curve has a cusp, what is happening here on the polar curve? Answer: You have a point of inflection.
- If the original curve has a point of inflection, what is happening here on the polar curve? Answer: You have a cusp.
- If the original curve has a point where the curve passes through twice, what is happening here on the polar curve? Answer: The polar curve has a tangent line with two points on it.
- If the center of the polarity conic lies on the original curve, what is happening here on the polar curve? Answer: The polar curve must be tangent to the line at infinity, like a parabola.