Philosophy of Mathematics
Worksheet #1

1) *Goldbach's Conjecture*

Choose any whole number less than 100. Find as many ways as possible to express your number as the sum of two prime numbers.
(E.g., If we choose 30, we can write it as 7+23, 11+19, or 13+17.)

What statements can you make about what you have discovered? Can you guess what Goldbach's Conjecture states?

2) *Difference of Two Squares*

Choose any whole number less than 50. Find as many ways as possible to express your number as the difference of two square numbers.
(E.g., If we choose 15, we can write it as $4^2 - 1^2$ or as $8^2 - 7^2$.)

What statements can you make about what you have discovered?

Philosophy of Mathematics
Worksheet #2

Set Name: **Novels** Description: Novels that Mr. York has read
Members: {*War and Peace, Moby Dick, Tale of Two Cities, Men of Mathematics*…}

Set Name: **Cities** Description: Cities with a population greater than 5 million.
Members: {Bombay, Sao Paulo, New York, Paris, Cairo, Tokyo…}

Set Name: **Prime** Description: Prime numbers
Members: {2, 3, 5, 7…}

Set Name: **Non-Prime** Description: Natural numbers that aren't prime
Members: {1, 4, 6, 8, 9…}

Set Name: **Two-Digit** Description: All two-digit whole numbers
Members: {10, 11, 12…}

Instructions: First of all, be sure that you understand the above sets. Each set below has the unusual characteristic that its members are sets. You need to fill in each set's members. To simplify matters somewhat, you are only required to consider all of the sets that are defined on this page. The first one has been done for you.

Set Name: **Number-Sets** Description: Sets that have numbers as members
Members: {Prime, Non-Prime, Two-Digit}

1) Set Name: **23-Sets** Description: Sets that include the number 23
Members: { }

2) Set Name: **X-Sets** Description: Sets that aren't a member of Number-Sets
Members: { }

3) Set Name: **N-Sets** Description: Sets having names beginning with "N"
Members: { }

4) Set Name: **A-Sets** Description: Sets having names beginning with "C"
Members: { }

5) Set Name: **S-Sets** Description: Sets that *are* members of themselves
Members: { }

6) Set Name: **R-Sets** (Russell's set!) Description: Sets that *are not* members of themselves
Members: { }

7) Is S-Sets a member of itself?

8) Is R-Sets a member of itself? (Russell's question!)

9) *The Liar's Paradox*. Is the below statement true or false?
"This statement is false."

10) *The Barber's Paradox*. Suppose there is a town with just one male barber; and that every man in the town keeps himself clean-shaven: some by shaving themselves, some by attending the barber. It seems reasonable to imagine that the barber obeys the following rule: He shaves all and only those men who do not shave themselves.
Does the barber shave himself?

Philosophy of Mathematics
Worksheet #3

1) *Fermat's Little Theorem*

Let X be equal to 2. Let Q be any whole number, but don't make it too big. Let $D = X^Q - X$.

Question: By just looking at the value of Q, how can we predict whether or not D will be evenly divisible by Q? (Choose a variety of possible values for X and Q.)

2) *Sum of Two Squares*

Choose any whole number less than 200. Find as many ways as possible to express your number as the sum of two square numbers.
(E.g., If we choose 170, we can write it as $1^2 + 13^2$ or as $7^2 + 11^2$.)

What statements can you make about what you have discovered? (You may look at the backside of this sheet to gain some insights.)

Philosophy of Mathematics
Worksheet #4

Background for Gödel's Proof

Table of Symbols within PM and *Gödel numbering*

Symbol	Gödel #	Meaning	Symbol	Gödel #	Meaning
~	1	not	(	8	left parenthesis
∨	2	or	)	9	right parenthesis
⊃	3	if…then…	,	10	comma
∃	4	there exists…	+	11	plus
=	5	equals	·	12	multiplication
0	6	zero	x	13	variable
s	7	successor of	y	17	variable
			z	19	variable

Note: ss0 stands for 2.

Vocabulary

PM: the given formal axiomatic system within which all the axioms, definitions, and rules are given. PM is assumed to be the system given in *Principia Mathematica* (by Russell and Whitehead), but could be any formal axiomatic system that includes the basic properties of arithmetic (e.g. addition, multiplication).

formula: a well-formed formula (wff) or statement within the system PM that is made in accordance to the rules. A formula may be either true or false. A formula becomes a *theorem* once it is proven. "x =" is not a proper formula. Examples of proper formulas include:

x = ssss0, which means "x equals 4"
sssss0 = ss0 + ss0, which means "5 equals 2 + 2"
(∃x)(x = ss0 + sss0), which means "there exists an x such that x equals 2+3"
(∃x)(x = ss0) ⊃ (∃y)(~(y = 0)), which means "If there exists x such that x equals 2, then there exists y such that y is not equal to 0."

proof: A sequence of formulas, each one of which is justified according to the rules and axioms, and the last formula in the proof is the theorem that is to be proven.

Gödel numbering:

- The key idea is that any possible formula or proof can be coded as a unique Gödel number, and any Gödel number can be decoded to get the formula or proof that it represents.
- To determine what a Gödel number represents, we break it down into its prime factorization.
- The bases of the prime factorization of any Gödel number are always consecutive prime numbers.
- The Gödel number of a single formula is such that each exponent in the prime factorization tells us a symbol in the formula.
- The Gödel number of a proof – or perhaps an entire system of proofs – is such that each exponent in the prime factorization is itself the Gödel number of a single formula.
- Example: The expression "(x)" has the Gödel number $2^8 \cdot 3^{13} \cdot 5^9 = 797{,}161{,}500{,}000{,}000$.

Exercises:

1) What is the formula represented by the Gödel number 31104000000? (Hint: the prime factorization is $2^{13} \cdot 3^5 \cdot 5^6$.)

2) What is the Gödel number for the formula "z = ss0 + y"? (Give your answer in prime factorization form.)

3) Consider the formula
(x=s0) ⊃ (∃y)(y=ssss0)
   a) Translate this formula into a statement.
   b) What is the Gödel number for the formula .

4) What is the Gödel number for the below proof? (Give your answer in prime factorization form.)
x = sss0
y = x
y = sss0

5) *Extra Credit!* What is the Gödel number for the whole of *Principia Mathematica*?

Prime Numbers up to 2000 (in groups of 250)

2	251	503	751	1009	1259	1511	1753
3	257	509	757	1013	1277	1523	1759
5	263	521	761	1019	1279	1531	1777
7	269	523	769	1021	1283	1543	1783
11	271	541	773	1031	1289	1549	1787
13	277	547	787	1033	1291	1553	1789
17	281	557	797	1039	1297	1559	1801
19	283	563	809	1049	1301	1567	1811
23	293	569	811	1051	1303	1571	1823
29	307	571	821	1061	1307	1579	1831
31	311	577	823	1063	1319	1583	1847
37	313	587	827	1069	1321	1597	1861
41	317	593	829	1087	1327	1601	1867
43	331	599	839	1091	1361	1607	1871
47	337	601	853	1093	1367	1609	1873
53	347	607	857	1097	1373	1613	1877
59	349	613	859	1103	1381	1619	1879
61	353	617	863	1109	1399	1621	1889
67	359	619	877	1117	1409	1627	1901
71	367	631	881	1123	1423	1637	1907
73	373	641	883	1129	1427	1657	1913
79	379	643	887	1151	1429	1663	1931
83	383	647	907	1153	1433	1667	1933
89	389	653	911	1163	1439	1669	1949
97	397	659	919	1171	1447	1693	1951
101	401	661	929	1181	1451	1697	1973
103	409	673	937	1187	1453	1699	1979
107	419	677	941	1193	1459	1709	1987
109	421	683	947	1201	1471	1721	1993
113	431	691	953	1213	1481	1723	1997
127	433	701	967	1217	1483	1733	1999
131	439	709	971	1223	1487	1741	
137	443	719	977	1229	1489	1747	
139	449	727	983	1231	1493		
149	457	733	991	1237	1499		
151	461	739	997	1249			
157	463	743					
163	467						
167	479						
173	487						
179	491						
181	499						
191							
193							
197							
199							
211							
223							
227							
229							
233							
239							
241							

Even Numbers as the **Sum of Two Primes**

4 = 2+2
6 = 3+3
8 = 3+5
10 = 3+7; 5+5
12 = 5+7
14 = 3+11; 7+7
16 = 3+13; 5+11
18 = 5+13; 7+11
20 = 3+17; 7+13
22 = 3+19; 5+17; 11+11
24 = 5+19; 7+17; 11+13
26 = 3+23; 7+19; 13+13
28 = 5+23; 11+17
30 = 7+23; 11+19; 13+17
32 = 3+29; 13+19
34 = 3+31; 5+29; 11+23; 17+17
36 = 5+31; 7+29; 13+23; 17+19
38 = 7+31; 19+19
40 = 3+37; 11+29; 17+23
42 = 5+37; 11+31; 13+29; 19+23
44 = 3+41; 7+37; 13+31
46 = 3+43; 5+41; 17+29; 23+23
48 = 5+43; 7+41; 11+37; 17+31; 19+29
50 = 3+47; 7+43; 13+37; 19+31
52 = 5+47; 11+41; 23+29
54 = 7+47; 11+43; 13+41; 17+37; 23+31
56 = 3+53; 13+43; 19+37
58 = 5+53; 11+47; 17+41; 29+29
60 = 7+53; 13+47; 17+43; 19+41; 23+37; 29+31
62 = 3+59; 19+43; 31+31
64 = 3+61; 5+59; 11+53; 17+47; 23+41
66 = 5+61; 7+59; 13+53; 19+47; 23+43; 29+37
68 = 7+61; 31+37
70 = 3+67; 11+59; 17+53; 23+47; 29+41
72 = 5+67; 11+61; 13+59; 19+53; 29+43; 31+41
74 = 3+71; 7+67; 13+61; 31+43; 37+37
76 = 3+73; 5+71; 17+59; 23+53; 29+47
78 = 5+73; 7+71; 11+67; 17+61; 19+59; 31+47; 37+41
80 = 7+73; 13+67; 19+61; 37+43
82 = 3+79; 11+71; 23+59; 29+53; 41+41
84 = 5+79; 11+73; 13+71; 17+67; 23+61; 31+53; 37+47; 41+43
86 = 3+83; 7+79; 13+73; 19+67; 43+43
88 = 5+83; 17+71; 29+59; 41+47
90 = 7+83; 11+79; 17+73; 19+71; 23+67; 29+61; 31+59; 37+53; 43+47
92 = 3+89; 13+79; 19+73; 31+61
94 = 5+89; 11+83; 23+71; 41+53; 47+47
96 = 7+89; 13+83; 17+79; 23+73; 29+67; 37+59; 43+53
98 = 19+79; 31+67; 37+61
100 = 3+97; 11+89; 17+83; 29+71; 41+59; 47+53

102 = 5+97; 13+89; 19+83; 23+79; 29+73; 31+71; 41+61; 43+59
104 = 3+101; 7+97; 31+73; 37+67; 43+61
106 = 3+103; 5+101; 17+89; 23+83; 47+59; 53+53
108 = 5+103; 7+101; 11+97; 19+89; 29+79; 37+71; 41+67; 47+61
110 = 3+107; 7+103; 13+97; 31+79; 37+73; 43+67
112 = 3+109; 5+107; 11+101; 23+89; 29+83; 41+71; 53+59
114 = 5+109; 7+107; 11+103; 13+101; 17+97; 31+83; 41+73; 43+71; 47+67; 53+61
116 = 3+113; 7+109; 13+103; 19+97; 37+79; 43+73
118 = 5+113; 11+107; 17+101; 29+89; 47+71; 59+59
120 = 7+113; 11+109; 13+107; 17+103; 19+101; 23+97; 31+89; 37+83; 41+79; 47+73; 53+67; 59+61
122 = 13+109; 19+103; 43+79; 61+61
124 = 11+113; 17+107; 23+101; 41+83; 53+71
126 = 13+113; 17+109; 19+107; 23+103; 29+97; 37+89; 43+83; 47+79; 53+73; 59+67
128 = 19+109; 31+97; 61+67
130 = 3+127; 17+113; 23+107; 29+101; 41+89; 47+83; 59+71
132 = 5+127; 19+113; 23+109; 29+103; 31+101; 43+89; 53+79; 59+73; 61+71
134 = 3+131; 7+127; 31+103; 37+97; 61+73; 67+67
136 = 5+131; 23+113; 29+107; 47+89; 53+83
138 = 7+131; 11+127; 29+109; 31+107; 37+101; 41+97; 59+79; 67+71
140 = 3+137; 13+127; 31+109; 37+103; 43+97; 61+79; 67+73
142 = 3+139; 5+137; 11+131; 29+113; 41+101; 53+89; 59+83; 71+71
144 = 5+139; 7+137; 13+131; 17+127; 31+113; 37+107; 41+103; 43+101; 47+97; 61+83; 71+73
146 = 7+139; 19+127; 37+109; 43+103; 67+79; 73+73
148 = 11+137; 17+131; 41+107; 47+101; 59+89
150 = 11+139; 13+137; 19+131; 23+127; 37+113; 41+109; 43+107; 47+103; 53+97; 61+89; 67+83; 71+79

Odd Numbers as the **Difference of Two Squares**

3 = $2^2 - 1^2$
5 = $3^2 - 2^2$
7 = $4^2 - 3^2$
9 = $5^2 - 4^2$
11 = $6^2 - 5^2$
13 = $7^2 - 6^2$
15 = $4^2 - 1^2$; $8^2 - 7^2$
17 = $9^2 - 8^2$
19 = $10^2 - 9^2$
21 = $5^2 - 2^2$; $11^2 - 10^2$
23 = $12^2 - 11^2$
25 = $13^2 - 12^2$
27 = $6^2 - 3^2$; $14^2 - 13^2$
29 = $15^2 - 14^2$
31 = $16^2 - 15^2$
33 = $7^2 - 4^2$; $17^2 - 16^2$
35 = $6^2 - 1^2$; $18^2 - 17^2$
37 = $19^2 - 18^2$
39 = $8^2 - 5^2$; $20^2 - 19^2$
41 = $21^2 - 20^2$
43 = $22^2 - 21^2$
45 = $7^2 - 2^2$; $9^2 - 6^2$; $23^2 - 22^2$
47 = $24^2 - 23^2$
49 = $25^2 - 24^2$
51 = $10^2 - 7^2$; $26^2 - 25^2$
53 = $27^2 - 26^2$
55 = $8^2 - 3^2$; $28^2 - 27^2$
57 = $11^2 - 8^2$; $29^2 - 28^2$
59 = $30^2 - 29^2$
61 = $31^2 - 30^2$
63 = $8^2 - 1^2$; $12^2 - 9^2$; $32^2 - 31^2$
65 = $9^2 - 4^2$; $33^2 - 32^2$
67 = $34^2 - 33^2$
69 = $13^2 - 10^2$; $35^2 - 34^2$
71 = $36^2 - 35^2$
73 = $37^2 - 36^2$
75 = $10^2 - 5^2$; $14^2 - 11^2$; $38^2 - 37^2$
77 = $9^2 - 2^2$; $39^2 - 38^2$
79 = $40^2 - 39^2$
81 = $15^2 - 12^2$; $41^2 - 40^2$
83 = $42^2 - 41^2$
85 = $11^2 - 6^2$; $43^2 - 42^2$
87 = $16^2 - 13^2$; $44^2 - 43^2$
89 = $45^2 - 44^2$
91 = $10^2 - 3^2$; $46^2 - 45^2$
93 = $17^2 - 14^2$; $47^2 - 46^2$
95 = $12^2 - 7^2$; $48^2 - 47^2$
97 = $49^2 - 48^2$
99 = $10^2 - 1^2$; $18^2 - 15^2$; $50^2 - 49^2$
101 = $51^2 - 50^2$
103 = $52^2 - 51^2$
105 = $11^2 - 4^2$; $13^2 - 8^2$; $19^2 - 16^2$; $53^2 - 52^2$
107 = $54^2 - 53^2$
109 = $55^2 - 54^2$
111 = $20^2 - 17^2$; $56^2 - 55^2$
113 = $57^2 - 56^2$
115 = $14^2 - 9^2$; $58^2 - 57^2$
117 = $11^2 - 2^2$; $21^2 - 18^2$; $59^2 - 58^2$
119 = $12^2 - 5^2$; $60^2 - 59^2$
121 = $61^2 - 60^2$
123 = $22^2 - 19^2$; $62^2 - 61^2$
125 = $15^2 - 10^2$; $63^2 - 62^2$
127 = $64^2 - 63^2$
129 = $23^2 - 20^2$; $65^2 - 64^2$
131 = $66^2 - 65^2$
133 = $13^2 - 6^2$; $67^2 - 66^2$
135 = $12^2 - 3^2$; $16^2 - 11^2$; $24^2 - 21^2$; $68^2 - 67^2$
137 = $69^2 - 68^2$
139 = $70^2 - 69^2$
141 = $25^2 - 22^2$; $71^2 - 70^2$
143 = $12^2 - 1^2$; $72^2 - 71^2$
145 = $17^2 - 12^2$; $73^2 - 72^2$
147 = $14^2 - 7^2$; $26^2 - 23^2$; $74^2 - 73^2$
149 = $75^2 - 74^2$
151 = $76^2 - 75^2$
153 = $13^2 - 4^2$; $27^2 - 24^2$; $77^2 - 76^2$
155 = $18^2 - 13^2$; $78^2 - 77^2$
157 = $79^2 - 78^2$
159 = $28^2 - 25^2$; $80^2 - 79^2$
161 = $15^2 - 8^2$; $81^2 - 80^2$
163 = $82^2 - 81^2$
165 = $13^2 - 2^2$; $19^2 - 14^2$; $29^2 - 26^2$; $83^2 - 82^2$
167 = $84^2 - 83^2$
169 = $85^2 - 84^2$
171 = $14^2 - 5^2$; $30^2 - 27^2$; $86^2 - 85^2$
173 = $87^2 - 86^2$
175 = $16^2 - 9^2$; $20^2 - 15^2$; $88^2 - 87^2$
177 = $31^2 - 28^2$; $89^2 - 88^2$
179 = $90^2 - 89^2$
181 = $91^2 - 90^2$
183 = $32^2 - 29^2$; $92^2 - 91^2$
185 = $21^2 - 16^2$; $93^2 - 92^2$
187 = $14^2 - 3^2$; $94^2 - 93^2$
189 = $15^2 - 6^2$; $17^2 - 10^2$; $33^2 - 30^2$; $95^2 - 94^2$
191 = $96^2 - 95^2$
193 = $97^2 - 96^2$
195 = $14^2 - 1^2$; $22^2 - 17^2$; $34^2 - 31^2$; $98^2 - 97^2$
197 = $99^2 - 98^2$
199 = $100^2 - 99^2$

Numbers as the **Sum of Two Squares**

(Numbers that are missing cannot be expressed as the sum of two squares.)

2 = $1^2 + 1^2$
5 = $1^2 + 2^2$
8 = $2^2 + 2^2$
10 = $1^2 + 3^2$
13 = $2^2 + 3^2$
17 = $1^2 + 4^2$
18 = $3^2 + 3^2$
20 = $2^2 + 4^2$
25 = $3^2 + 4^2$
26 = $1^2 + 5^2$
29 = $2^2 + 5^2$
32 = $4^2 + 4^2$
34 = $3^2 + 5^2$
37 = $1^2 + 6^2$
40 = $2^2 + 6^2$
41 = $4^2 + 5^2$
45 = $3^2 + 6^2$
50 = $1^2 + 7^2$; $5^2 + 5^2$
52 = $4^2 + 6^2$
53 = $2^2 + 7^2$
58 = $3^2 + 7^2$
61 = $5^2 + 6^2$
65 = $1^2 + 8^2$; $4^2 + 7^2$
68 = $2^2 + 8^2$
72 = $6^2 + 6^2$
73 = $3^2 + 8^2$
74 = $5^2 + 7^2$
80 = $4^2 + 8^2$
82 = $1^2 + 9^2$
85 = $2^2 + 9^2$; $6^2 + 7^2$
89 = $5^2 + 8^2$
90 = $3^2 + 9^2$
97 = $4^2 + 9^2$
98 = $7^2 + 7^2$
100 = $6^2 + 8^2$
101 = $1^2 + 10^2$
104 = $2^2 + 10^2$
106 = $5^2 + 9^2$
109 = $3^2 + 10^2$
113 = $7^2 + 8^2$
116 = $4^2 + 10^2$
117 = $6^2 + 9^2$
122 = $1^2 + 11^2$
125 = $2^2 + 11^2$; $5^2 + 10^2$
128 = $8^2 + 8^2$
130 = $3^2 + 11^2$; $7^2 + 9^2$
136 = $6^2 + 10^2$
137 = $4^2 + 11^2$
145 = $1^2 + 12^2$; $8^2 + 9^2$
146 = $5^2 + 11^2$
148 = $2^2 + 12^2$
149 = $7^2 + 10^2$
153 = $3^2 + 12^2$
157 = $6^2 + 11^2$
160 = $4^2 + 12^2$
162 = $9^2 + 9^2$
164 = $8^2 + 10^2$
169 = $5^2 + 12^2$
170 = $1^2 + 13^2$; $7^2 + 11^2$
173 = $2^2 + 13^2$
178 = $3^2 + 13^2$
180 = $6^2 + 12^2$
181 = $9^2 + 10^2$
185 = $4^2 + 13^2$; $8^2 + 11^2$
193 = $7^2 + 12^2$
194 = $5^2 + 13^2$
197 = $1^2 + 14^2$
200 = $2^2 + 14^2$; $10^2 + 10^2$
202 = $9^2 + 11^2$
205 = $3^2 + 14^2$; $6^2 + 13^2$
208 = $8^2 + 12^2$
212 = $4^2 + 14^2$
218 = $7^2 + 13^2$
221 = $5^2 + 14^2$; $10^2 + 11^2$
225 = $9^2 + 12^2$
226 = $1^2 + 15^2$
229 = $2^2 + 15^2$
232 = $6^2 + 14^2$
233 = $8^2 + 13^2$
234 = $3^2 + 15^2$
241 = $4^2 + 15^2$
242 = $11^2 + 11^2$
244 = $10^2 + 12^2$
245 = $7^2 + 14^2$
250 = $5^2 + 15^2$; $9^2 + 13^2$
257 = $1^2 + 16^2$
260 = $2^2 + 16^2$; $8^2 + 14^2$
261 = $6^2 + 15^2$
265 = $3^2 + 16^2$; $11^2 + 12^2$
269 = $10^2 + 13^2$
272 = $4^2 + 16^2$
274 = $7^2 + 15^2$
277 = $9^2 + 14^2$
281 = $5^2 + 16^2$
288 = $12^2 + 12^2$
289 = $8^2 + 15^2$
290 = $1^2 + 17^2$; $11^2 + 13^2$
292 = $6^2 + 16^2$
293 = $2^2 + 17^2$
296 = $10^2 + 14^2$
298 = $3^2 + 17^2$
305 = $4^2 + 17^2$; $7^2 + 16^2$
306 = $9^2 + 15^2$
313 = $12^2 + 13^2$
314 = $5^2 + 17^2$
317 = $11^2 + 14^2$
320 = $8^2 + 16^2$
325 = $1^2 + 18^2$; $6^2 + 17^2$; $10^2 + 15^2$
328 = $2^2 + 18^2$
333 = $3^2 + 18^2$
337 = $9^2 + 16^2$
338 = $7^2 + 17^2$; $13^2 + 13^2$
340 = $4^2 + 18^2$; $12^2 + 14^2$
346 = $11^2 + 15^2$
349 = $5^2 + 18^2$
353 = $8^2 + 17^2$
356 = $10^2 + 16^2$
360 = $6^2 + 18^2$
362 = $1^2 + 19^2$
365 = $2^2 + 19^2$; $13^2 + 14^2$
369 = $12^2 + 15^2$
370 = $3^2 + 19^2$; $9^2 + 17^2$
373 = $7^2 + 18^2$
377 = $4^2 + 19^2$; $11^2 + 16^2$
386 = $5^2 + 19^2$
388 = $8^2 + 18^2$
389 = $10^2 + 17^2$
392 = $14^2 + 14^2$
394 = $13^2 + 15^2$
397 = $6^2 + 19^2$
400 = $12^2 + 16^2$
401 = $1^2 + 20^2$
404 = $2^2 + 20^2$
405 = $9^2 + 18^2$
409 = $3^2 + 20^2$
410 = $7^2 + 19^2$; $11^2 + 17^2$
416 = $4^2 + 20^2$
421 = $14^2 + 15^2$
424 = $10^2 + 18^2$
425 = $5^2 + 20^2$; $8^2 + 19^2$; $13^2 + 16^2$
433 = $12^2 + 17^2$
436 = $6^2 + 20^2$
442 = $1^2 + 21^2$; $9^2 + 19^2$

The first number that can be expressed in 4 ways is…
1105 = $4^2 + 33^2$; $9^2 + 32^2$;
$12^2 + 31^2$; $23^2 + 24^2$