Lesson Plans for (10th Grade Main Lesson) (last updated Nov 2014)

Greek Geometry and Deductive Proofs

Overview

This course traces the evolution of Greek thought in mathematics, from the school of Pythagoras up until the works of Euclid, and the brilliant mind of Archimedes. To a large degree, this entails a history of mathematical proofs, which aids the students as they learn the art of deductive reasoning. The students are exposed to a variety of different styles of proofs, including visual proofs, indirect proofs, and Euclidean proofs.

At the start of the course, the students begin to answer the question, "What is a proof?" by studying six different proofs of the Pythagorean Theorem and the Pythagorean proof that the $\sqrt{2}$ is irrational. This leads naturally to the question, "What is a *good* proof?", and then, finally, to the idea of an axiomatic system of proofs. Much attention is then given to Euclid's 13-volume work, *The Elements*, and its impact on the world of mathematics. We thoroughly study *Book I* from this work, which consists of 48 theorems and culminates in Euclid's proof of the Pythagorean Theorem.

This main lesson block falls at the start of the tenth grade year and is intended to be an introduction to deductive, Euclidean-style proofs. Further work with proofs is continued in the tenth grade geometry track class.

Notes for the Teacher

- *Warm-ups*. Years ago, for this main lesson, I always did began each class with a musical offering, where a student would perform a (non-amplified) instrumental piece or sing a song. Then we did the *Golden Verses of Pythagoras* as the verse each morning (two paragraphs per morning), and the last week we recited *Euclid Alone has Looked on Beauty Bare*, by Edna St. Vincent Millay. I no longer spend the time in my main lesson doing this warm-up because we now do adequate warm-ups together as a whole high school.
- *Main Lesson Books*.
  - I have the students do a fair bit of writing for this block, but, as always, I don't want to burden the students with too much homework.
  - In general, for this block, I say that there should be a minimum of two essays per week (each about 500 words). Students hoping for an "A" should be doing one or two extra essays per week, as well.
  - This is a summary of the writing assignments that I give for this block:
    - **Week #1:** Required: (1) The "three-student" essay (see day #1); (2) "School of Pythagoras"
      Optional: Biography of Pythagoras; "Irrational Numbers";
      "Mystery of the Square and the proof that square root of two is irrational"
      Various Euclidean constructions (from the backside of sheet #1;
      Greek Thought (including "Wisdom vs. Intelligence")
    - **Week #2:** Required: (1) "Six proofs of the Pythagorean Theorem";
      (2) "Proofs" essay, addressing such questions as: The Greek search for truth; Why do we need to prove anything? What is a proof? What is a good proof? A perfect proof? The best-possible proof?
      Optional: Expansion of either of the two required essays; construction of 17-gon
      Area Transformations: problems from Constructions Sheet #2
    - **Week #3:** Required: Essay on "The Elements"; Introduction to the main lesson book
      Optional: making the essay on "The Elements" into a double essay;
      Epilogue; various proof summaries;
  - Quotes
    - "There is no royal road to geometry." Reply given when the ruler Ptolemy I Soter asked Euclid if there was a shorter road to learning geometry than through Euclid's *Elements*.
    - The sign above Plato's Academy entrance read: "Let no-one ignorant of geometry enter here".
    - "At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. I had not imagined there was anything so delicious in the world."
      -- Russell Bertrand

Day #1

- I can remember three times in my life having my thinking transformed: 7th grade algebra; 10th grade geometric proofs; projective geometry (11th grade in Waldorf schools).
- This block is not just math. It's writing, history, philosophy, music. Be open!
- What this course will be about: *experiencing* the most revolutionary thoughts of the greatest Greek mathematicians. So we will need to live into what it was like to be Greek 2500 years ago.
- Briefly introduce the idea of the Pythagorean School (but don't give it all away!). Mention the four basic subjects that you had to be proficient in: Arithmetic, Geometry, Astronomy, Music.

Group Work: How would it be different to be a Pythagorean student, pre-Greek student or modern student (general student, and math in particular)? Possible answers:

Pre-Greek	**Pythagoreans**	**Study Today**
Mythology/Religion as a basis	Always creating new math	Learning old stuff
Math was empirically-based (business, construction).	very little to go on.	Practical
	Philosophical	Nobody debating
No need for proofs	Everything was debatable	"Why do I need to prove it?"
Constructing perfection	Proofs were crucial	Nobody knows 10% of it
Schooling was practical	You could learn most of it	Algebra based
	Geometry based	Calculators
	Don't care about measuring.	Fast communication/media
	God's laws of universe.	
	Slow communication	

Note: The idea of "education for all" has changed dramatically over the years. In 1890 fewer than 7% of the 14 year-olds in the United States were enrolled in high school, with roughly half of those going on to graduate. High school and beyond were reserved for the elite, with fewer students graduating from high school back then than earn Masters and Ph.D. degrees today. By the beginning of World War II, 49% graduated high school. Today, about 88% of adults are high school graduates, 32% have Bachelor's degrees, and 3% have a Doctorate and/or professional degree.

If time allows: Begin *Euclidean Constructions Sheet #1* in groups (It may be sufficient to simply write the problems on the board.)

- Hand out sheet, *Basic Euclidean Constructions* (ideally this should have been gone over in track class last week along with the front side of the *Constructions Sheet #1*).
- Have them work on the backside, which includes these problems: quadrisecting an angle; trisecting an angle; finding center of a circle; construction of 12-gon, 10-gon, 15-gon, 9-gon, 7-gon, 17-gon. (Note: each "-gon" is assumed to be "regular".)
- Mention that some of these are *very* difficult.

Hanging questions:

1) How do you think that the Pythagoreans knew that the Pythagorean Theorem was true?
2) How can I say that something is definitely true?
3) What do you think the basic principles were for the Pythagoreans that explain the universe (i.e., what was the basis for their worldview)?

Homework:

- Read: (1) Expectations; (2) Golden Verses of Pythagoras

Day #2

- Go over Course Expectations sheet. What are good essays. Good main lesson book.

Historical Background

- Earlier civilizations had advanced math and geometry, but had no proofs. It was the Greeks who had first developed proofs.
- Earlier civilizations were more interested in using math for practical purposes.
- Pythagoras was the first to call himself a philosopher, which means "lover of wisdom", but for him really meant a "seeker of the truth".
- For the Greeks, the important questions were not *"What is true?"*, but rather *"Why is it true?"*, or *"How can say that this is definitely true?"*.
- The Greeks were not interested in practical mathematics. Proofs were the most important part of mathematics, and this was truly a philosophical endeavor.

<u>The School of Pythagoras:</u>

- Hanging question #3:
  - Each culture has their basis for understanding the universe (a worldview).
  - Number was the Pythagoreans basis for their worldview.
  - What do you think our basis is for our worldview? (Perhaps leave this hanging until next year with the Descartes block.)
- Go over hanging question #1 and #2: The Pythagoreans were among the first to need a proof. They questioned what they believed; they questioned their thinking.

<u>The Great Greek Geometric Game</u>

- The difference between using the guess and check method to get the best possible pentagon, and using the official pentagon construction to get a *theoretically perfect* pentagon.
- Give the rules: (1) a straight-edge can only be used to draw a line between two given points. (2) a compass can be set to the distance between two given points, and be used to draw part of a circle using a given setting and with a given point as the center.

<u>Group Work</u>: Continue *Euclidean Construction Sheet #1*, especially finding circle's center.

<u>If time allows</u>: Introduce commensurability (See day#3)

<u>Hanging questions</u>: Challenge problems from *Euclidean Construction Sheet #1*, especially trisecting an angle, and finding the center of a circle.

Day #3

Pythagoras and his School

- Go over the sheet "Pythagoras and his school". (Don't hand it out.)
- The word "harmony" originally means joining or fitting together; "art" has the same root meaning. For the Greeks, beauty, harmony and proportion were closely related ideas.
- Beauty is also tied to mathematics. Something is beautiful if it exhibits an inner harmony or wholeness that arises from its parts joining together in proper proportion. Give examples of how this is reflected in art, architecture, the beauty of the human being, and in music (e.g., the story of Pythagoras walking past a blacksmith's shop). All of this was seen as connected to the mathematical idea of ratio.

Commensurability:

- The Pythagorean principle that "Numbers Rule the Universe" was at the heart of the Pythagorean belief system. They believed that God had created numbers with perfection, beauty and simplicity. This central principle leads to the following ideas:
    - The ratio of any two numbers can be expressed as the ratio of two whole numbers.
    - Any two lengths are commensurable (i.e. we can find an exact ratio, in whole number form, of the two lengths).
    - Any number can be expressed as a fraction.
- If I have two lines that are of length A and B (the longer one), then we can think of commensurability in two ways: (1) How can I make it such that a certain number of A lines would be equal to a certain number of B lines? (2) B is how much of A? (e.g., perhaps B is $^4/_3$ of A)
- Examples: Try to answer the above questions with these values for A and B:
    - 12 and 20 (answers: 5A = 3B, and B = $^5/_3$ A)
    - 2⅓ and 5½ (answers: 33A = 14B, and B = $^{33}/_{14}$ A)
    - $2\frac{4}{15}$ and $3\frac{7}{10}$ (answers: 111A = 68B, and B = $^{111}/_{68}$ A)
- *Explain why the square was a mystery. (Most important!)*
    - If a rectangle has dimensions 1 by ¾ then we know the ratio is 4:3.
    - We know that if we assign the side of the square to 1, then the diagonal is $\sqrt{2}$.
    - As with the above rectangle, if we knew what $\sqrt{2}$ was equal to as a fraction, then we would know the exact ratio of the square (diagonal:side). For example if $\sqrt{2}$ were equal to $^7/_5$ (which is close), then we could say that a square with side equal to 1 would have a ratio of diagonal : side = 7:5.
    - So, the question has now become: "What is the exact value of $\sqrt{2}$ as a fraction?"
- Here are some approximations for $\sqrt{2}$: $\frac{7}{5}, \frac{17}{12}, \frac{140}{99}, \frac{1393}{985}, \frac{8119}{5741}, \frac{114243}{80782}$
- Here are some fractional approximations for π:
    - $\frac{22}{7} = 3^1/_7$ (Archimedes' upper-bound); $\frac{223}{71} = 3\frac{10}{71}$ (Archimedes' lower-bound)
    - $\frac{355}{113} = 3\frac{16}{113}$ was known by the Chinese mathematician, Zu Chongzhi in the 5th century.

      This is incredibly accurate – a better one can't be found until $\frac{52163}{16604}$
- Mention that someone solved this mystery – but don't give away that it was proved to be impossible. Tell them that tomorrow we'll see the proof, and they will need to be extra focused.

- *Euclidean Constructions Sheet #1*.
    - Review the rules of the Great Greek Geometric Game.
    - Have students share their solutions.
    - Go over last problem (how to find the center of the circle). Emphasize why constructing a square around the perimeter won't work (you can't draw a line tangent to the circle without knowing the center).
    - Doing extra examples from this sheet in your main lesson book will count as extra credit. I'm especially interested to see your efforts on some of the harder ones.
- How the Greeks viewed the Pythagorean Theorem differently than we do today.

Hanging questions: Still leave hanging some problems from *Euclidean Construction Sheet #1*.

Day #4

- Remind everyone about footnoting in order to avoid plagiarism.

Thorough review!!

Give bio of Pythagoras. Πυθαγόρας (See notes at the end)

Mystery of the Square (continued)

- Review especially how the question of finding the ratio of the diagonal to the side led to the question of what the $\sqrt{2}$ is (exactly!) as a fraction.
- Before beginning the proof, mention that although the Pythagoreans wouldn't have used algebra, we will use algebra to help us understand the same exact thought process.
- It may be best [for the teacher] to pretend to be the Pythagorean student that actually presented the proof, and pretend that the students are in the Pythagorean School.
- **Give Proof that $\sqrt{2}$ is irrational.** (Give a quick break before doing this.)
    - This proof is an indirect proof (or proof by contradiction).
    1. Assume that $\sqrt{2}$ can be expressed *exactly* as a *reduced* fraction. Let this fraction be $\frac{X}{Y}$, where X and Y are both whole numbers and have no common factor.
    2. Squaring both sides of $\frac{X}{Y} = \sqrt{2}$ we get: $\frac{X^2}{Y^2} = 2$, which becomes $X^2 = 2Y^2$
    3. Whether Y is even or odd, $2Y^2$ is must be even. Therefore X^2 is even, and also X must be even.
    4. Since $\frac{X}{Y}$ is a reduced fraction, Y must be odd.
    5. Since X is even, half of X (call it P) must be a whole number. $X = 2P$.
    6. Since X = 2P, we can substitute 2P in for X with the equation given in step 2. So we get $(2P)^2 = 2Y^2 \rightarrow 4P^2 = 2Y^2 \rightarrow 2P^2 = Y^2$.
    7. $2P^2$ is even, therefore Y^2 is even, and also Y must be even.
    8. Step 4 and 7 are in contradiction of each other. Our assumption must therefore be false.
    9. *$\sqrt{2}$ cannot be expressed as a fraction. (And it then follows that the side and the diagonal of a square cannot be expressed as a whole number ratio; they are not commensurable).*
- The consequences of this proof:
    - The person who leaked this discovery is believed to have been drowned.
    - Through thinking they were able to show that one of their core beliefs was flawed.
    - This leaves us wondering: What other things that appear to be true, aren't?

If time allows:

- Review the *Shear and Stretch*
- Do Pythagorean Theorem proof #1 and #2 (From Six Proofs of the Pythagorean Theorem).

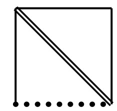
Day #5

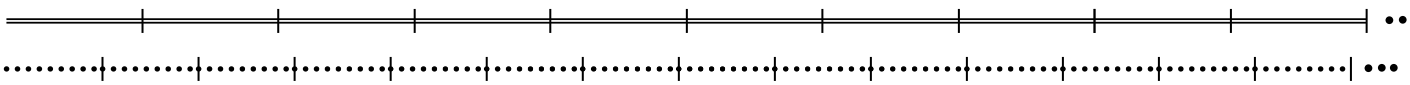
<u>Review</u>

- Especially the proof that $\sqrt{2}$ is irrational.
- Thorough Review of week – perhaps some time is needed to catch up and tie loose ends together.

<u>What does it mean to *not* be commensurable</u>?

- If the length of the square's diagonal is copied side-by-side infinitely many times, and the square's side is also copied side-by-side infinitely many times, then there will *never* be a place where the two arrive at the same exact spot. (See below.)





<u>The key ideas of irrational numbers</u>:

- An irrational number cannot be expressed as an exact fraction, or as an unending or repeating decimal. (Recall that any "normal" fraction is *exactly* equal to a decimal that either repeats or ends, and vice-versa.)
- *An irrational number does have an exact value, but you cannot say exactly what it is.*
- A calculator gives an approximation of any irrational number, as would a computer that could calculate its value to a million decimal places.
- We can do exact calculations with irrational numbers. (e.g. $3\sqrt{2} \cdot 5\sqrt{2} = 30$)
- π and $\sqrt{2}$ are irrational numbers. How do we know for sure that they are actually irrational. Perhaps they would end up repeating or ending after 2 million decimal places. How do we know this doesn't happen? Answer: there is a proof that it's impossible!

<u>The Six Proofs of the Pythagorean Theorem:</u>

- Go over proofs #1 and #2.
- In groups: Try to figure out proofs #3, #4.
- Answer the following: Draw a circle with its diameter. Choose any point (try 4 or five different ones) on the circle and then draw a line from each of the two ends of the diameter to the chosen point on the circle. What mathematical law can be stated from this situation? (Answer: This is the Theorem of Thales)

<u>Hanging questions</u>:

- How do proofs #5 and #6 work? (Give hints.)

Day #6

- Handout first week main book grades. Give "main lesson book pep talk". (See below.)

<u>The three famous Greek construction problems</u>

- *Some of the questions and mysteries that the greatest minds in history struggled with:*
    - There was a huge debate about why the planets moved in such a strange way across the heavens. Many great thinkers died convinced that the sun goes around the earth.
    - Many of the greatest thinkers died thinking that their method for solving one of the construction mysteries was correct, or not knowing which constructions were possible. You will not!
    - The three famous construction problems were:
        - The trisection of an angle.
        - The squaring of a circle (construction of a square with area equal to a given circle).
        - The doubling of a cube (how to determine the length of the edge of a cube that has twice the volume of a given cube).
    - The construction of these polygons were also a puzzle to them:
      7-gon, 9-gon, 11-gon, 13-gon, 17-gon, 19-gon, etc.
    - Tell which ones on Euclidean Constructions Sheet #1 were impossible.

<u>The Six Proofs of the Pythagorean Theorem:</u>

- In 1940, Elisha Loomis published his book *The Pythagorean Proposition*, in which he gives 370 proofs of the Pythagorean Theorem, including one by Leonardo da Vinci, and one by James Garfield (U.S. President in 1881).
- Lead students through proofs #5 and #6.

<u>Group work.</u> Discuss the following questions:

- In your opinion, which of the six proofs is the best one? What makes it the best one?
- What are the strengths and weaknesses of each proof?

<u>How the Greeks changed the world through thought</u>.

- Question: Why were the Greeks so interested in math? Answer: they were interested in the search for truth, and math embodied truth better than other subjects.
- A quote from a student: "In Greece, Pythagoras gained intelligence; in Egypt he gained wisdom."
- List on the board (with class input) all the difference between wisdom (e.g., intuitive) and intelligence (e.g., comes through your thinking process).
- Image of Pythagoras in Egypt: A couple of Egyptian priests debating about who had made the better right triangle. Pythagoras saying to himself, “Who cares. I just want to know what the mathematical laws are for right triangles.”
- The emergence of philosophers as opposed to a sage or "wise man/woman".
- How can the world be explained? The transition from mythology/religion to philosophy – developing a logical system of thought.
- The Greeks were the first to systematically think about and question their own thinking.
- The transition from the "what" to the "why".
- How do I know what I know? How can I know what is true with any certainty?
- The Greeks were the first to realize that you couldn’t always trust that your perceptions and thoughts were true. The Greeks were searching for truth. Thus came the need for proof.
- The Greeks introduced deductive reasoning, which has had a huge impact on how we think today.

<u>Hanging questions</u>:

- What makes for a good proof?
- What would a perfect proof be?

Day #7

Review:

- What were the great mysteries and puzzles?
- Why were the Greeks so interested in math?
- Now tell the story of Gauss's discovery of the 17-gon construction. Hand out his solution to it.
  - In 1796, at the age of 19, Carl Friedrich Gauss proved that all regular n-gons where $n = 2^{(2^n)}+1$ are also constructible. By letting n = 0, 1, 2, 3, 4 we get polygons with 3, 5, 17, 257, and 65537 sides respectively. This discovery supposedly convinced Gauss that he should dedicate his life to mathematics.
  - Gauss did not provide a method for constructing the 17-gon. In 1800, Johannes Erchinger provided the first method for doing this.
  - In a paper he wrote published in 1801, Gauss listed all of 37 possible regular polygons under 300 that can be constructed by Euclidean methods. These are:
    - 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, 20, 24, 30, 32, 34, 40, 48, 51, 60, 64, 68, 80, 85, 96, 102, 120, 128, 136, 160, 170, 192, 204, 240, 255, 256, 257, 272
    - Perhaps it is best to only give a few of these and leave it as a challenge for the students to derive the rest.

History Overview (Give out *Greek Math Timeline*)

- From the beginning to the end of the Greek period is about 1700 years. From the Renaissance to the present is about 700 years.
- From Pythagoras to Archimedes is about the same time span as from Galileo to Einstein.
- There were hundreds of mathematicians/philosophers (Zeno, Hippocrates, Eudoxus, etc.) in the 200 years between Pythagoras and Euclid.
- Emphasize the teacher/student sequence from Socrates to Plato to Aristotle to Alexander. Then comes the founding of Alexandria and Euclid.
- Emphasize that Euclid was the "great organizer".

Group work

- Have students try to prove the X Theorem (vertical angle theorem) and the Y Theorem (supplementary angle theorem).
- *Transformation of Area problems* (if there is time). Try to determine how you can transform a quadrilateral to a triangle of equal area by using the Shear and Stretch.

The Six Proofs (cont.) (Class Discussion) Answer the hanging questions…

- Have all 6 proofs on the board.
- Which of the six proofs is the best one?
- What are the assumptions that each proof makes?
- What makes for a good proof? (Ans: a good proof should be (1) easy to understand and (2) unquestionable. There seems to be a trade-off between these two qualities.)
- What would a perfect proof be? (Ans: a proof that makes no assumptions; but this is impossible!)
- What would be the best possible proof? (Hopefully they see that it would be one which has undisputable assumptions.)
- Which of the 6 proofs would Euclid have liked? (Ans: none of them – they all assume too much!) Mention that Euclid wrote a book with 465 theorems (and proofs!) in it.

Hanging questions: What do you think Euclid's first theorem and 5 simple assumptions were?
What do you think his proof of the Pythagorean theorem was?

Day #8

- Quickly review all 6 proofs and their assumptions.
  - Review the idea that none of our 6 proofs would have been good enough for Euclid. So the question we are left to think about is: What did Euclid come up with for a best-possible proof of the Pythagorean theorem?
- Ideas for "Proofs essay":
  - The Greek search for truth.
  - Why do we need to prove anything? What is a proof?
  - What is a good proof? A perfect proof? The best-possible proof?
- Ask he students: What was so hard about the proofs for the X and Y theorem?

<u>An introduction to *The Elements*</u>.

- Euclid as the great organizer.
- *Euclid* gathered (nearly) all of the math known at that time and put it into one book, called *The Elements*, which consisted of 13 chapters, called Book 1, II, etc.
- The book proves a total of 465 theorems.
- *Euclid's 5 Postulates*. He begins the whole book with 5 postulates, which are assumptions about geometry. He then proves his first theorem based upon those postulates, and then proves the second theorem based upon the first theorem and the 5 postulates. The power of these postulates is that they are very simple and unquestionable.
- The hanging questions from yesterday: Tell what his first theorem was. Give his 5 postulates, which are assumptions about geometry. (But don't hand out sheet on *Foundations of Euclid's Elements* until tomorrow.)
- Hand out *Summary of Book I.*

<u>Group Work</u>:

- Tell that they aren't responsible for understanding #2, 7, 21, 39, 40 – these theorems are used primarily as stepping stones to prove other, more important theorems.
- Tell the students what these terms mean: subtend, included angle, exterior angle.
- For each theorem do the following:
  - Find which theorem is (these are listed in order of appearance): SAS, Isos Δ Th., SSS, Vertical Angle Th (X th.), Supplementary Angle Th. (Y Th.), ASA, Alternate Interior Angle Th. (Z Th.), Shear and Stretch, Pythag. Th. Also find all converses.
  - For the rest of the theorems, label them as one of the following:
    - * meaning "I've seen this before."
    - \# meaning "This is new, but I understand it."
    - ? meaning "I don't understand this one."

<u>Hanging questions</u> (for what will be done in class tomorrow):

- How do you think he proved I-15? (Vertical Angle Theorem)
- How did he prove his first theorem, I-1? (construction of an equilateral triangle)

Day #9

More on *The Elements*.

- In reality, very few improvements have been made on *The Elements*. It remained *the* geometry textbook for more than 2000 years.
- Euclid wanted to prove simple theorems (e.g. such as the X theorem) and more complicated theorems (e.g. the Pythagorean Theorem, and how to construct a 15-gon) all based upon the simplest possible assumptions.
- The idea of an ***axiomatic system***.
    - The proof of each theorem is based upon the 5 postulates and any previously proven theorems.
    - The four possible ways to justify a step in a proof: Postulate, Common Notion, Definition, Previously proven Theorem
    - What are the differences between:
        - A theorem (a mathematical statement that has been proven)
        - A postulate (assumption about geometry; a mathematical statement that is accepted without proof.)
        - A common notion (a law about logic).
- Handout and briefly go over these sheets: *Foundation of The Elements; Summary of the 13 Books*

Give Euclid's proof of Th. I-15(*Vertical angle theorem*).

- After giving the proof, see if the class can come up with the justification of each step.
- Q.E.D stands for *quod erat demonstrandum.*
- Point out that since this proof (I-15) relies on I-13, we are left wondering how he proves that one.

Give Euclid's proof of Th. I-1(*Construction of an equilateral triangle*).

- Emphasize that for the proof of Th. I-1 Euclid needs to do more than just construct an equilateral triangle – he needs to prove that it's an equilateral triangle.
- After giving the proof, see if the class can come up with the justification of each step.

Group work (if there is time). Try to figure out each of these. Don't give answers until tomorrow.

- The transformation of a quadrilateral into a triangle of equal area.
- The transformation of a quadrilateral into a rectangle of equal area.
- The transformation of a rectangle into a square.

Hanging questions (for what will be done in class tomorrow):

- How do you think he proved I-9 (Bisecting an angle)?
- How do you think he proved I-13 (Supplementary angle theorem)?
- Why did he word the fifth postulate as he did?

Day #10

<u>Continuing with *The Elements*</u>.

- Have you noticed that this has been all about geometry? Math was based upon geometry for them. *The Elements* was not just about geometry – it was all of math. That is different than now.
- Review: The four possible ways to justify a step in a proof: Postulate, Common Notion, Definition, Previously proven Theorem
- From Euclid to modern times mathematicians have been on a quest for simpler and simpler postulates upon which all of math could be based.
- In reality, Euclid probably created *the Elements* backwards. Imagine that he starts with a proof of the Pythagorean Theorem, then he asks what assumptions are being made (SAS ≅ Th, converse of Y Th, converse of Z Th, shear and stretch, construction of square, parallel lines). Then he sets out to prove each of these, and asks what the new assumptions are, and then proves those assumptions, etc. The amazing thing is that he finds instead getting more and more assumptions, he finds that many of the proofs end up using the same basic assumptions. He goes further and further until he is left with just 5 basic, irrefutable assumptions.
- Review the difficulty of proving the X and Y theorems (they seem so obviously true!). Point out that Euclid proves these as his 13^{th} and 15^{th} theorems. The first few proofs (e.g., I-5) are harder because you don't have as much to draw on.
- Make sure that everyone understands all the "required" theorems of Book I. Note those theorems that nobody in the group understands.
- Why he worded the fifth postulate as he did. He avoided the words "infinite, never". He avoided the word "parallel" as much as possible – it doesn't appear in the first 26 theorems.
- Handout *Selected Proofs* sheet.

<u>Give Euclid's proof of Th. I-13</u>(*Supplementary angle theorem*).

- After giving the proof, see if the class can come up with the justification of each step.

<u>Give Euclid's proof of Th. I-9</u>(*Construction of an angle bisector*).

- After giving the proof, see if the class can come up with the justification of each step.

<u>Transformation of Area problems</u> (if there is time.)

- Show how to do the following (which was started in groups yesterday):
    - The transformation of a quadrilateral into a triangle of equal area.
    - The transformation of a quadrilateral into a rectangle of equal area.
    - The transformation of a rectangle into a square. (Leave explanation as a hanging question.)
- Explain how to do the transformation of an n-gon (n>4) into a square of equal area. This involves going from an n-gon to a quadrilateral to a trapezoid to a rectangle to a square.
- Start work on *Euclidean Constructions Sheet #2*. (In groups.)

<u>Hanging questions</u> (for what will be done in class tomorrow):

- How do you think he proved I-5? (Hint: use I-3 and I-4)
- How do you think he proved I-32 (triangle Interior Angle Theorem)?

Day #11

- Talk about how the ML Book should include Intro and Epilogue and that their title page should include their favorite quote from the Pythagorean Verses.
- Check to see if there are more questions regarding Book I theorems.

Discussion comparing proofs of th I-5 and th I-15.

- The proof of I-5 needed to be very clever because he had little to draw on. The proof of I-15 was much more straight forward because he had a fair bit to draw on.

Give Euclid's proof of Th. I-5 (*Isosceles Triangle Theorem*).

- After giving the proof, see if the class can come up with the justification of each step.

Give Euclid's proof of Th. I-32 (*Triangle Interior Angle Theorem*).

- After giving the proof, see if the class can come up with the justification of each step.
- Note now he should have had an "angle addition postulate".

Criticism of Euclid. People have had Euclid under a microscope for around 2000 years. While most of his work remains solid, there are some criticism:

- Euclid's idea of a line had finite length. The modern idea is that lines are infinitely long.
- Not all definitions were included (e.g. "bisect"). Others can't be defined (e.g. "point", "line").
- He unconsciously made some assumptions that should have been listed as additional postulates. For example, he assumes some things by looking at drawings. For example, there should be an angle addition postulate (as in Th. I-32) and a segment addition postulate.
- His 5th postulate was too complicated. Mathematicians tried for nearly 2000 years to prove the 5th postulate, but without any success. Note that he avoided mentioning the word "parallel" until Th. I-27, and didn't use the 5th postulate until Th. I-29.

Transformation of Area (if there is time):

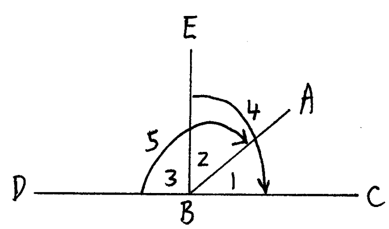
- See if anyone was able to figure out why the transformation from a rectangle to a square is true.
- (If necessary) *Finish Euclidean Constructions Sheet #2*. (In groups)

Different styles of proofs:

- *Paragraph.* This was the style that Euclid used (and that most modern mathematicians use today). Show the Greek original of Th. X-1.
- *Steps with Justifications.* This is the style that I have used for this block. I've put it into this form to make it easier for students to understand.
- *Two-Column.* This is the style used by many high school geometry textbooks.

- The Proof for I-13 is perhaps the hardest to summarize. It goes like this:

  Initially, we are given ∠1 and ∠5 such that they are adjacent and form a straight line. We then construct line EB perpendicular to DC, thereby creating two right angles, ∠3 and ∠4. Euclid says that the sum of these two right angles is equal to the sum of ∠1, ∠2 and ∠3. He then says that the sum of ∠1, ∠2 and ∠3 is also equal to the sum of ∠5 and ∠1. Therefore ∠5 and ∠1 is equal to the sum of ∠3 and ∠4, which is two right angles.



Hanging questions (for what will be done in class tomorrow):

- Th. I-5 relies on I-4. How do you think he proved I-4? (SAS Δ ≅ Th.)
- Th. I-32 relies on I-29. How do you think he proved I-29 (F, Z, and C Theorems)?

Day #12

Review

- Ask students to give a short, verbal summary of I-5.
- Ask: I-4 (SAS $\Delta \cong$ Th.) is the major theorem that is used to prove I-5. What did he prove I-4? Answer: He only used CN4! He says that if you place one triangle on top of the other (where the two have equal SAS), such that the tips of the equal angles are on top of one another, and one of the sides is placed on top of the other equal side, then we can say that the other known equal sides will lie on top of each other, and so will the other end points of the two pairs of known equal sides. This means that the two triangles must be congruent, and that the remaining corresponding side, and the two pairs of remaining corresponding angles must also be equal.

Give Euclid's proof of Th. I-29 (which is the Alternate Interior Angle (Z) Theorem, the Corresponding Angle (F) Theorem, and the Same-Side Interior Angle (C) Theorem.)

- Th. I-29 is the first time that Euclid uses the 5^{th} postulate. Theorem I-27 is the first time he uses the word "parallel".
- After this, I-29 is used to prove many other theorems, which is why we can say that most of the of *The Elements* depends upon Postulate 5.
- Have the students justify the steps of I-29.

Summary of Archimedes

- Stories about Archimedes
  - Story of the king's crown, and "Eureka".
  - Story of his death.
- Some of Archimedes' key Accomplishments:
  - Law of the Lever (a brief review from 7^{th} grade).
  - Archimedes' Principle (Law of floating bodies)
  - First steps of integral calculus.
  - Archimedes' Ratio for Volumes: Cone : Sphere : Cylinder = 1 : 2 : 3
  - Archimedes' method for calculating π.
  - Mechanical inventions
    - Parabolic mirror
    - Archimedean Screw
    - Pulley system to move ships
  - Area of a parabolic segment is $^4/_3$ the area of the triangle having the same base and equal height.
  - The surface area of a sphere is four times greater than the area of the greatest circle on it.
  - The area of a circle is equal to the area of a right triangle that has legs equal to the radius and the circumference of the circle.
  - The surface of any right circular cylinder is equal to the area of the circle whose radius is equal to the geometric mean of the height and diameter of the cylinder.

Hanging questions (for what will be done in class tomorrow):

- Tomorrow is the big day! We finally get to see Euclid's proof of the Pythagorean Th (I-47). How do you think he did it?

Day #13

Euclid's proof of the Pythagorean Theorem (Th. I-47)

- Try to give the full presentation is 30 minutes or less.

Group Work:

- Do review sheet.

Hanging questions (for what will be done in class tomorrow):

- What is π? It answers the question: For any circle, how many times longer is the circumference than its diameter?
- How do you think Archimedes calculated a value for π?

Day #14

Archimedes Calculation of π. (This lesson can be done as a general outline in ½ hour. A complete presentation on the details would take a few days of track class.)

- Answer the previous hanging question: π can only be determined exactly through thinking; a computer program would need a systematic logical method for determining π.
- Give an outline of Archimedes' method.
  - Explain his general idea of having inscribed and circumscribed n-gons about a circle that has a diameter of exactly one, and therefore a circumference of exactly π.
  - For now only calculate the perimeter of the inscribed (which is 3) and circumscribed 6-gon (which is $2\sqrt{3} \approx 3.46$). Then give the following results, but mention that (1) How he amazingly got from the perimeter of one polygon to the next polygon with twice the number of sides takes much more explanation than what we are doing here; (2) Archimedes used fractions instead of decimals.

For a 6-gon:	$3.0 < \pi < 3.4641016$
For a 12-gon:	$3.1058285 < \pi < 3.2153903$
For a 24-gon:	$3.1326286 < \pi < 3.1596599$
For a 48-gon:	$3.1393502 < \pi < 3.1460862$
For a 96-gon:	$3.1410320 < \pi < 3.1427146$

- There was then a contest to calculate π to as many digits as possible
  - The first person to reach 100 digits was John Machin (British) in 1706.
  - In 1844, Johann Dase (German) calculated π to 200 places.
  - In 1855, Richter calculates π to 500 places.
  - In 1873, William Shanks (British) calculated π to 707 places, but in 1945 (63 years after Shanks' death) it was discovered (by Daniel Ferguson) that the 527th place was incorrect.
  - In 1946, Daniel Ferguson calculated π to 620 decimal places – the last record achieved without the aid of a calculating device.

Go over review sheet.

- Give time to work on ML books at the end of class.

Day #15

TEST!!

Biography of Πυθαγόρας (Pythagoras)

One day (ca. 570 b.c.) a merchant from Samos named Mnesarchus was traveling and found himself in Delphi. The oracle at Delphi was an older woman of blameless life chosen from among the peasants of the area. The oracle would fall into a trance, allowing Apollo to possess her spirit. In this state she prophesied. Before making the long dangerous journey home, Mnesarchus decided to consult with the oracle about when the best time would be to begin his journey home. Instead, she told him that his wife was about to give birth to a boy, and this son would grow up to "exceed all others in beauty and wisdom."

As a young man, Pythagoras had three main teachers who influenced his interest in mathematics, geometry, astronomy and music, one of which was Thales, who was an old man by then. Thales advised Pythagoras to travel to Egypt to learn more. When Pythagoras was around 35 years old, he traveled to Egypt. Pythagoras was refused admission to all the temples except the one where he was accepted into the priesthood after completing the rites necessary for admission. This was unheard of for a foreigner to be accepted into the Egyptian priesthood.

The Persian empire had quickly expanded to include much of Central Asia, Babylonia, the Middle East. Egypt was next; when Pythagoras was about 45 years old, Persia invaded Egypt. Pythagoras was taken to Babylon as a prisoner. While he was there he was instructed in their sacred rites, and learned about their arithmetic, music and the other mathematical sciences taught by the Babylonians.

At the age of 50, Pythagoras was released as a prisoner from Babylon and returned to Samos, which was now also overtaken by the Persians. Pythagoras tried to form a school, but it was not accepted by the people there, so after two years, he left Samos and went to Croton, in southern Italy. He quickly attained extensive influence in Croton, and many people began to follow him. He gave many eloquent speeches, which led many of the people of Croton to abandon their luxurious and corrupt way of life and to devote themselves to the purer system which he introduced.

Many people saw Pythagoras as a divine figure, sent by the gods to benefit humankind. His school was something of a mixture between a brotherhood, a college, a monastery, and a commune. It was based upon the religious teachings of Pythagoras and was very secretive. The members were bound by a secret vow to Pythagoras and to each other. In terms of mathematics, the School of Pythagoras was most famous for its two important discoveries: The Pythagorean Theorem and irrational numbers. In their ethical practices the Pythagorean were famous for their mutual friendship, unselfishness, and honesty.

See next page, titled: "Pythagoras and his School".

When Pythagoras was nearly 100 years old, Cylon, who was a wealthy nobleman from Croton, was eager to join the School. However, Pythagoras rejected Cylon because he was said to be of ill character. It is believed that Cylon incited others in Croton who were also envious of the Pythagoreans, and that they attacked the School. Pythagoras and other of his followers fled and took refuge in a building outside of town. The attackers burned the building; many inside, perhaps including Pythagoras himself, died in the flames.

The school thrived under Pythagoras's leadership for almost half a century. Certainly, the Pythagorean Society thrived for many years after Pythagoras's death, and spread from Croton to many other cities. Pythagoras had a tremendous influence on many of the mathematical and philosophical thinkers that came later, especially Plato. However, above all else, Pythagoras was famous for leaving behind him a way of life, and a new philosophical way of looking at the world.

Pythagoras and his School

Quotes

- “Every man has been made by God in order to acquire knowledge and contemplate.”
- “Numbers rule the universe.”
- “Were it not for number and its nature, nothing that exists would be clear to anyone. You can observe the power of number not only in the affairs of the gods, but in all the acts and thoughts of people.”

Aphorisms

- Walk on unfrequented paths.
- Govern your tongue before all other things, following the gods.
- Do not poke the fire with a sword.

Their Practices

- Communal living
- Egalitarian; included women
- Strict diet
- Strict secrecy

The Four Subjects of Study

- *Arithmetic*: Number in Itself.
- *Geometry*: Number in Space
- *Music*: Number in Time
- *Astronomy*: Number in the Heavens

Central Beliefs

- That at its deepest level, reality is mathematical in nature.
- That philosophy can be used for spiritual purification.
- That the soul can rise to union with the divine.
- That certain symbols have mystical significance.

The Characteristics of Numbers (Listed here is just one of several characteristics for each.)

- *One*: Harmony
- *Two*: Balance
- *Three*: Wisdom
- *Four*: Justice
- *Five*: Nature
- *Six*: Perfection
- *Seven*: Fate
- *Eight*: Love
- *Nine*: Compassion
- *Ten*: God
- *Even numbers*: Feminine
- *Odd numbers*: Masculine