Lesson Plans for (12th Grade Main Lesson)
Introduction to Calculus (last updated Nov 2011)

Overview

To some degree, this course is a continuation from last year's Descartes main lesson on the theme of the evolution of consciousness. In that vain, this main lesson is a blending of history, philosophy, writing and mathematics – although there is more of an emphasis on mathematics with this main lesson than there was with the Descartes block.

For many of the students, this course will be the climax of their mathematical studies, even though they study more math topics in their track classes during the remainder of the year. I usually teach this Calculus main lesson in October, which gives time in the track classes to prepare for the main lesson. In my school, further work is done in the advanced math track class with calculus skills after the main lesson block is completed.

Notes for the Teacher

- This block was written with the intention of it being 16 days, with the last two days being review and test. It can be done in fewer days, but the teacher will need to carefully decide which topics to leave out.
- As opposed to the material intended for the track class (where in my school the class is split into faster and slower-paced sections), the material for this main lesson is carefully designed so that all students can follow it without getting lost or feeling overwhelmed. If done well, this can be an empowering experience even (or especially) for the students that have struggled for years with math.
- All Discovery Sheets are intended to be done in groups during class – not for homework. The students do close to one sheet per day. **If time is short, then sheets #1 and #9 may largely be skipped.**
- This may be a good block to have the students do a journal instead of a traditional main lesson book.
- I like to put a famous quote on the board before the start of each class.
- Prerequisites. Before the course begins, it is important that the students are at least familiar with (1) series and basic sigma notation, (2) basic function notation (e.g., given $f(x) = x^2 + 5$, find $f(x-6)$).

Day #1

- Note: Most of "Day #1" can be covered in just a few minutes if the block is less than 16 days.
- Course Expectations. Including the goal to not leave anyone behind. Daily review will be essential!
- Math groups need to be formed for the whole block. Each group should have students of equal ability, rather than have weak students and stronger students together.
- Historical Background.
  - The Greeks. To some degree, the roots of calculus can be said to originate with the ancient Greeks.
    - Some of Archimedes' work essentially uses calculus concepts.
    - Zeno (ca. 450BC) posed eight paradoxes of motion, three of which are given below.
      - Note for the Teacher:
        - These paradoxes can be covered in detail, or they can be only mentioned superficially if the amount of time given to the block is limited.
        - If there is adequate time, just state the paradoxes on the first day. Then have students in groups discuss: "Where is the flaw in the argument?" Further discussion will happen tomorrow.

      1. *The Dichotomy*. "*That which is in motion must arrive at the half-way point before it arrives at the goal.*" (Aristotle, *Physics* VI:9) I have seen this interpreted in two ways. One way is that you can never get anywhere because you must first pass the halfway point, then, from there, pass the remaining halfway point, etc. This requires you to complete an infinite number of tasks, which Zeno maintains is an impossibility. Therefore you can never completely arriving at your destination. Motion is impossible!
      2. *Achilles and the Tortoise*. "*In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead.*" (Aristotle, *Physics* VI:9) Thus, Achilles arrives first at the tortoise's first point, then arrives at the tortoise's next point, etc. Thus, at each moment Achilles reaches a location where the tortoise has already been, but he still has farther to go. There are an infinite number of points Achilles must reach where the tortoise has already been, so he will never overtake the tortoise.

3. <u>*The Arrow*</u>. "*If everything when it occupies an equal space is at rest, and if that which is in motion is always occupying such a space at any moment, the flying arrow is therefore motionless.*" (Aristotle, *Physics* VI:9) Since time is a series of indivisible units, and at any given instant the arrow is at rest, then throughout all time the arrow is at rest. Once again, motion cannot happen.

- <u>Hanging Question</u>: What was Zeno really trying to prove?
- **Do Discovery Sheet #1 (*Series Formulas*).** The purpose of this sheet, to some degree, is to prove that the series ½ + ¼ + $^1/_8$ +… = 1 (exactly). This arises as a question out of Zeno's dichotomy paradox. If the time in this course is limited, then sheet #1 can be skipped. A couple of the questions deal with the idea of what happens to the value of an expression if an exponent goes to infinity. This essentially is the idea of a limit, which will be further pursued tomorrow.

Day #2

- <u>Further Discussion of the Paradoxes</u>
  - Zeno of Elea created these paradoxes to support Parmenides' philosophy that "all is one" and, contrary to what our senses tell us, the belief in plurality and change is mistaken – that motion is nothing but an illusion.
  - <u>Note</u>: With each of the paradoxes, often a student will say something like, "But I can <u>see</u> that Achilles passes the tortoise. That therefore proves that Zeno is wrong." Actually, Zeno's point is the opposite. He would say that what you see is not a proof at all. In fact, he says it demonstrates his point: his argument soundly proves that motion is impossible, which shows that what we see is an illusion.
- <u>The Idea of a Limit</u>:
  - We know that $0.\overline{3} = {}^1/_3$. This can be written as $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$, or $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{10^i} = {}^1/_3$
  - Similarly, we can say that $0.\overline{9} = 1$.
  - On Discovery Sheet #1, we saw that with the relation $y = x^n$, if $0 \le x < 1$ as n approaches ∞, x^n approaches zero. In other words, if n goes to infinity x^n gets infinitely close to y, which we will say is actually equal to zero.
  - We can write the above statement in limit notation as: Given $0 \le x < 1$ $\lim_{n\to\infty} x^n = 0$
  - We can start with the *Power Series Formula* $\sum_{i=0}^{n} x^i = \frac{x^{n+1} - 1}{x - 1}$ and then use the above limit to derive the

    the *Infinite Power Series Formula*, which states if $0 \le x < 1$, then $\lim_{n\to\infty} \sum_{i=0}^{n} x^i = \frac{1}{1 - x}$
  - We can now use the above *Infinite Power Series Formula* to resolve Zeno's dichotomy

    $1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = 2$, which can also be written as $\lim_{n\to\infty} \sum_{i=0}^{n} (½)^i = 2$

    Subtracting 1 from both sides gives us Zeno's dichotomy:

    $\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots = 1$, which can also be written as $\lim_{n\to\infty} \sum_{i=1}^{n} (½)^i = 1$
  - Give example limit problems, including one where the limit does not exist, such as…

    If $f(x) = \frac{x^2 + 10x + 21}{x + 3}$ we know that the domain does not include x = –3, but a limit allows us <u>sometimes</u> to dance around this problem. It may be helpful to show the graph (on Geometer's Sketchpad) of each.

    $\lim_{x \to -3} \frac{x^2 + 10x + 21}{x + 3} = 4$ $\quad\quad$ $\lim_{x \to 2} \frac{x^2}{x - 2}$ does not exist!

- There are two ways to do the above problem: I can plug in values very close to –3 (either slightly less than –3 or slightly more than –3), and look at my results – is it getting closer and closer to some value? Or I can do some algebra (e.g., factoring) to see if somehow I can make the fraction disappear.

- **Do Discovery Sheet #2 (*Limits & Average Speed*).**
  - Beforehand, give the students two average speed formulas:

$$r = \frac{\Delta d}{\Delta t} \quad \text{(Note: } \Delta \text{ means "change in")} \qquad\qquad r = \frac{d_2 - d_1}{t_2 - t_1}$$

  - This sheet has the students practice taking limits of expressions, and also has them work with average speed. It also hints toward instantaneous speed.
- Hanging Question: Draw attention to the question on Sheet #2 that asks: "What is your speed if you go 0 miles in 0 hours?" This is essentially asking what is 0÷0. We know that 0÷n=0, and n÷0→∞, so, "What is 0÷0?"

Day #3

- Review
  - The idea of a limit.
    - **Our "Limit Postulate":** (Note: The handling of the concept of limit that is being done here is fairly informal. Most college calculus textbooks handle limits in a more thorough and formal manner.) *"When evaluating an expression, if the result is known to get infinitely close to a certain number, then we say that it is exactly equal to that number."*
    - Note that the first calculus textbook (written by Marquis de L'Hospital, who was a student of Leibniz and Johann Bernoulli) included a postulate that stated two quantities differing by an infinitesimal are considered to be equal to each other – i.e., they are both equal and not equal at the same time.
  - The distinction between the imagined and the concrete. Using Zeno's dichotomy as an example, in the imagination, the sequence really can go on forever, so the sequence really can reach its limit of 1. In the concrete world, forever and infinity are not reachable by humans, so the limit is the unreachable.
  - Review the "limit postulate", again emphasizing that in a "real" calculus course limits are treated in a much more formal manner.
  - Give examples where the limit doesn't exist.
    - A rational expression that can't be reduced. (See previous day.)
    - A function that oscillates – e.g., limit $x \to \infty$ of $\sin(x)$.
  - Review the average speed formulas
  - Hanging question from yesterday ("What is 0÷0?"). This is essentially the *Arrow Paradox*. 12÷2 asks "How many 2's add to get 12?" So 0÷0 could be anything.
- Instantaneous Speed.
  - The question on Sheet #2 that asks: "What is your speed if you go 0 miles in 0 hours?" is essentially asking for instantaneous speed.
  - Ask: "What is instantaneous speed?", and "How are average speed and instantaneous speed different?"
    - Be clear that with average speed we are recording the distance and time between two points. For example, the average speed of a car going between two points could be 40mph. It might be that the car went exactly 40mph the whole time, or that the speed varied greatly between those two points. With instantaneous speed we are interested in the exact speed at a particular location or instant.
    - A car's speedometer shows instantaneous speed.
    - We can "sense" instantaneous speed.
    - Average speed is what has been done in school up until calculus.
- **Do Discovery Sheet #3 (*Galileo's Experiment*).** Notes for the teacher:
  - Normally Galileo's Law of Falling Bodies is given as $d = 4.9t^2$. They will try to derive this relationship between distance and time (although with a factor different than 4.9) on the Discovery Sheet, so don't give it to them until after they have done the sheet. This law is true for freefall, neglecting any friction (air resistance), where t is in seconds and d is in meters. In reality, Galileo did not drop anything from the top of the Tower if Pisa, but rather rolled balls down an inclined plane. This slowed things down and made timing easier. The conclusion from using inclined planes or freefall should theoretically be the same: *the change in distance is directly proportional to the square of the change in time*.

- This experiment may have been done for real in 10th grade mechanics. Even so, it is still a valuable review to do the Discovery Sheet here, in 12th grade.
- The purpose of this Discovery Sheet is to try to figure out the formula that relates distance and time (which is what Galileo did), and also to think further about how we would calculate instantaneous speed (which Galileo didn't do). Seeing lawfulness from the numbers in the table is a hard, but valuable, exercise.
- Since the angle of inclination of the inclined plane is 10°, the acceleration of the ball would be sin(10°) times g, where g (the gravitational acceleration constant) is 9.8 m/s^2, which is 1.703 m/s^2.
  Normally, for freefalling bodies, the constant of proportionality is 4.9, which is half of 9.8 (the gravitational acceleration). Therefore, the constant of proportionality for the inclined plane is half of 1.703, which is where the 0.851 comes from. Thus, the distance formula for this inclined plane (which is one of the questions on this Discovery Sheet) is $d = 0.851 \cdot t^2$.

- Hanging Questions: (1) Question#1 from the Discovery Sheet: "What conclusions can you reach about the relationship of distance and time by looking at the data?"
  (2) For our purposes, what is the advantage of having a formula to calculate distance?
  (3) Wouldn't it be nice to also have a formula for calculating instantaneous speed?

Day #4

- Review
  - Hanging Question#1: What is the relationship between time and distance in the data?
    - If the time increases by a factor of n, the distance increases by a factor of n^2.
    - At any moment, $d \div t^2$ is equal to the same constant, k (e.g., 0.851). $d_2 : t_2^2 = d_1 : t_1^2$
    - Similarly, we can say that the distance is directly proportional to the square of the time. Written in ratio form, this becomes $d_2 : d_1 = t_2^2 : t_1^2$
  - Hanging Question#2: We can get exact values without measuring. This enables us to calculate the distances for time intervals that are as close together as we please.
  - Make sure that the students understand what the formula $d(t) = k \cdot t^2$ really means.
    - "k" is the *constant of proportionality*. This value of k varies depending on the steepness of the inclined plane. A steeper plane means that the distance accelerates more quickly, which is indicated by a greater value of k.
    - What is the difference between these formulas? (Put a table on the board for each one of the following three functions.)
      - $d(t) = 4.9 \cdot t^2$ (this is the formula for freefall)
      - $d(t) = 0.851 \cdot t^2$ (this is the formula if the inclined plane has an angle of 10°.)
      - $d(t) = 3 \cdot t^2$ (this is the formula if the inclined plane is about 37.8°.)
      - We will be mostly using the last of these formulas because it is the easiest.
    - Using $3 \cdot t^2$, what does $d(10) = 300$ mean?
  - Average speed formulas. Go over the four average speed formulas that appear on sheet #4, especially the *Calculus Average Speed Formula*, which will allow us now to calculate instantaneous speed in an exact mathematical way.
- **Do Discovery Sheet #4 (*Instantaneous Speed, Part I*)**, up until the graphing problems.
- Graphing Polynomial Equations. (Practice problems appear on sheets #3,5,7)
  Give the general idea, including.
  - The number of bumps:
    - A first-degree linear equation (which is a line) has no bumps.
    - A second-degree quadratic equation (which is a parabola) has one bump.
    - A third-degree cubic equation has two bumps, and is the shape of a curvy "N".
    - A fourth-degree quartic equation has three bumps, and is the shape of a curvy "W".
  - A negative coefficient flips everything upside-down.
  - The bumps might merge together. Examples: $f(x) = x^3$ or $f(x) = x^4 - 2x^3$.
- **Finish Discovery Sheet #4**, which is the graphing problems.
- Hanging Question: How can we find a better method for calculating instantaneous speed?

Day #5

Review:

- Briefly review the basic forms of the graphs of polynomial functions.
- Make sure everyone understands the *Calculus Average Speed Formula*

$$\boxed{r = \frac{d(t+h) - d(t)}{h}}$$

Where is this going?

- On yesterday's sheet, in order to get an instantaneous speed at some time, t, we simply calculated an average speed closer and closer to t, until we could clearly see what the value was approaching.
- Many were then able to conclude that, for the case of $d(t) = 3t^2$, that all we had to do to find the instantaneous speed was multiply by 6 – in other words, the equation for instantaneous speed must be $v = 6 \cdot t$. This is correct. But, we are left wondering why this is correct. Is there a more mathematical way to derive this equation?
- We want to derive formulas for instantaneous speed that: (1) make sense; (2) make our work easier; (3) allow us to get around the problem of 0÷0.
- How do we get around the problem of 0÷0? Recall problem #3b on Sheet#2:

  $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$ Here we can factor and cancel, which simplifies the problem to:

  $\lim_{x \to 3} (x+3)$ And now we put in 3 without problem, giving an answer of 6.

- The ultimate useful formula is called a derivative, which allows us to determine the *instantaneous rate of change* of any function.
  - Note that in calculus we think of speed as the "change in distance *with respect to* time".
- **Do Discovery Sheet #5 (*Instantaneous Speed, Part II*)**
  - On this sheet we derive four formulas, each one getting more general, and more useful!

Day #6

Review:

- Since this is Monday, it is good to do a review of the whole of last week.
- Go over the four formulas from sheet #5. Be clear what each variable stands for, including "r" for average speed, and "v" for instantaneous speed (velocity).
- **Catch-up Day!**

Challenge Problem (This could engage a student needing a challenge for several days!)

- Derive summation formulas for Σi, Σi^2, Σi^3, Σi^4. (See sheet #7 for answers. Proofs in *HS Source Book*.)

Hanging Question:

- "What do we get if we add together infinitely many, infinitely small things?"

Day #7

- Review. (It may be good to have what is below clearly written on the board.)
  - How can we calculate instantaneous speed, for example using the distance formula $d(t) = 3t^2$ at $t = 4$?
    - Method #1. The Method of *Getting really close!*
      - Use the average speed formula, and plug in two values really close to t=4. For example (with sheet#4, prob#1g), if t=4 and t+h=4.0001, then the average speed is 24.0003.
    - Method #2. Finding a formula for instantaneous speed. These are the steps:
      - Find a formula for average speed between any two time values, t and t+h. This is the average speed formula $r = 6t + 3h$. Ask how we got this.
      - **The big step!!!** We can now just put 0 in for h to get the instantaneous speed formula $v(t) = 6t$.
      - Now we can just plug t=4 into this formula.
- **Do Discovery Sheet #6 (*The Derivative, Part I*)**
  - Emphasize that the last problem (finding the area under a curve) is our next big step. It introduces the second branch of calculus – integration.
  - If there is adequate time, it may be best to spend much of the class working on this last problem. Otherwise, more time can be spent on this last problem when they do groupwork tomorrow on Sheet #7.
  - Depending on how far you get, it may be best to tell the students (at the end of class) that the way calculus handles this is by adding together infinitely many infinitely thin rectangles.

The Definition of the Derivative. (Do at the end of class in preparation for sheet #7.)

- "Three important formulas":

1. The *Calculus Average Speed Formula* $\quad r = \dfrac{d(t+h) - d(t)}{h}$
2. The *Calculus Instantaneous Speed Formula* $\quad v(t) = \lim\limits_{h \to 0} \dfrac{d(t+h) - d(t)}{h}$
3. *The Derivative* (good for any instantaneous rate of change)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Note that each formula above is really a set of instructions that allows us to generate a formula for a specific purpose.
- Emphasize that the derivative doesn't just deal with distance and velocity (instantaneous rate of change of distance), but it works for any function (not just distance) and its instantaneous rate of change.

Important Note:

Be sure that for more complicated functions both f(x+h) and f(x) are put into the definition of the derivative *in parentheses* in the numerator.

- For example: given $f(x) = x^2 - 5x + 6$, we get $f(x+h) = x^2 + 2xh + h^2 - 5x - 5h + 6$
  Therefore using the definition of the derivative we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x^2 + 2xh + h^2 - 5x - 5h + 6) - (x^2 - 5x + 6)}{h}, \text{ etc.}$$

Hanging Question:

- Depending on how far the groups got on the last problem of sheet #6, one of these two questions can be left hanging: (1) How can we find an exact method for calculating the area under a curve; (2) How can we break the area under the curve into some arbitrary number (e.g., 7) of rectangles, and then use that knowledge to help us deal with infinitely many rectangles?

Day #8

Review

- Area Under the Curve, and the last problem from sheet #6. Mention that we will return to it on sheet #8.
- Have students share their ideas for finding the area under the curve (the last problem from sheet #6).
- The "Three Important Formulas" that lead to the definition of the derivative.
  - Show how the definition of the derivative can be used to find f '(x) of any function, such as f(x) = 8x2. Show all of the steps!!
  - Be sure to mention that there is often a shortcut way to find the derivative.
- Introduce the summation formulas.

$$\sum_{k=1}^{n} k = \tfrac{1}{2}n^2 + \tfrac{1}{2}n; \quad \sum_{k=1}^{n} k^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n; \quad \sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2$$

Groupwork

- Do Discovery Sheet #7 (The Derivative, Part II)
- The first problem (finding the derivative of f(x) = x3 by using the definition) should be started on the board, and then continued in groups. However, be sure that first they can FOIL out (x+10)3
- Do more work on the last problem of sheet #6 (finding area under the curve), if it is not yet finished.
- Hanging Question: For a function, f(x), what is the significance of the derivative, f '(x), in terms of the graph of f(x)?

Day #9

Review:

- Review the definition of the derivative.
- Go over, in detail, sheet #7, problem #1b (which is to determine the derivative of $f(x) = x^2 + 7x$ by using the definition of the derivative). At the end, emphasize that there are two ways to find the derivative: (1) Using the definition; (2) using a short-cut.
- Perhaps, also find the derivative of a really nasty function (worked out on the board) like $f(x) = x^4 + 5x^2 – 8x + 3$.
- Write down the derivative short-cuts:
  - *The Power Rule*. If $f(x) = k\cdot x^n$ then $\frac{dy}{dx} = nk\cdot x^{n-1}$
  - *The Addition Rule*. When taking the derivative of a multi-term polynomial function, you can simply take the derivative of each term separately.
  - *The Constant Rule*. The derivative of a constant is zero, because it isn't changing!
- Do derivative example: "What is the instantaneous speed at t = 5, given $d(t) = t^3 – 6t^2 + 9t + 2$" Ans: 24
- Review summation formulas.
- Remind everyone that just as we found an easy way to find the derivative of a function, so we will now (on sheet #8) find an easy way to derive a formula for finding the area under a curve.
- Review the idea of the area under the curve problem that appeared at the end of sheet #6.
- **Do Discovery Sheet #8 (*The Integral, Part I*)**
  - This is a difficult Discovery Sheet. Be sure to leave plenty of time to do it. It will likely need to be spread out over two days. The class may need a fair bit of guidance.
  - See the page "The Evolution of the Integral" for a summary of the key ideas from this sheet.
  - For starters, be sure that they understand what the coordinates mean for any function, f(x). (See below.)

Day #10

Review:

- Review the definition of the derivative.
- (Summation Formulas can be reviewed tomorrow!)
- Review the derivative short-cuts.
- There are many different notations for the derivative. We can say $f'(x)$, $\frac{dy}{dx}$, $\frac{d(f(x))}{dx}$
- Instead of saying "Find the derivative of $f(x) = x^4 - 3x^2$", we can say: "Find $\frac{d(x^4-3x^2)}{dx}$"

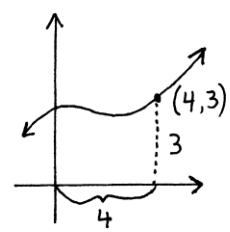
- **Finish Discovery Sheet #8 (*The Integral, Part I*)**
  - The answer to #6 (finding a general formula for the area under the curve):
    n = number of rectangles; we are finding the area under $f(x) = x^2$ from x = 0 to a.

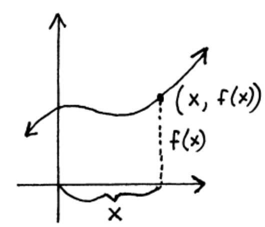
$$1^2\left(\frac{a}{n}\right)^3 + 2^2\left(\frac{a}{n}\right)^3 + 3^2\left(\frac{a}{n}\right)^3 + \ldots + n^2\left(\frac{a}{n}\right)^3 \rightarrow \left(\frac{a}{n}\right)^3\left(1^2 + 2^2 + 3^2 + \ldots + n^2\right)$$

$$\left(\frac{a}{n}\right)^3\left(\tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\right) \rightarrow \frac{a^3}{3} + \frac{a^3}{2n} + \frac{a^3}{6n^2}$$

  - The answer to #7c: If we let the number of rectangles go toward ∞, then we get $\tfrac{1}{3}a^3$.
  - **Note:** The results of this sheet are summarized on the handout "Evolution of the Integral".

- Hanging Question: On Sheet #5, we were supposed to graph $f(x) = x^3 + x^2 - 6x$. The problem is that we didn't get the peaks and valleys correct. How can we do this?





Day #11

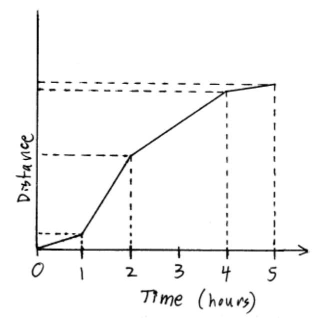
Review: (Most of the review of sheet #8 is summarized on the page "The Evolution of the Integral".)

- On Sheet #8, we found a formula for the area under the curve $f(x)=x^2$ from x =0 to a, by using n rectangles. That formula was: $\text{Area} = \frac{a^3}{3} + \frac{a^3}{2n} + \frac{a^3}{6n^2}$
- Then we took the limit of this formula as n goes to infinity, and got ⅓a^3. In calculus terminology this is: $\int_0^a f(x)dx = \frac{a^3}{3}$
- This is called the *integral* of f(x), and is used to find the area under $f(x)=x^2$ from x = 0 to a.
  - Example: Find the area under the curve $f(x) = x^2$ from x = 0 to 6. (Answer: 72.)
  - **Note: Be careful about saying that the integral gives us the area under the curve from x = 0 to some value a, for any function, f(x). This will only be true as long as long as F(0) = 0, which is true if f(x) is a polynomial function, but not true with some other functions, such as exponential functions and some trigonometric functions.**
- We can then use the formula $\int_0^a f(x)\,dx = ⅓a^3$ to calculate the area under the curve from x = 0 to a.

  For example, the area from x = 0 to 6 is 72.
  - All of the above is summarized below. Make sure that the students understand the following:

    $$\int_0^a f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\cdot\Delta x$$ (Note: The area of the k^{th} rectangle is $f(x_k)\cdot\Delta x$.)

    - Be sure to ask what happens as n goes to infinity.
- Derivative notation. There are different ways to express a derivative, as shown below:
  - Given $f(x) = x^3 + 6x^2$ or $y = x^3 + 6x^2$, we can ask Find $f'(x)$ or Find $\frac{dy}{dx}$
- **Do Discovery Sheet #9.** (If time is short, skip this sheet, but summarized the key concepts from it.)
- What does the Derivative mean in terms of the graph of f(x)?



  - Ask the students to create a graph of distance vs. time from the following scenario: *Jeff drives from home along a straight road. He drives rather slowly for the first hour, then drives an hour at a fast rate, followed by two hours at a medium speed, and ends by driving very slowly for one hour.*
  - The students should then see that speed at any moment is represented as the steepness at that point in the graph.
  - With a distance-time function (e.g., $d(t) = 3t^2$), the derivative gives us the instantaneous rate at which distance is changing with respect to time. We can see this as the slope of the curve.
  - With any function, f(x), the derivative gives us the instantaneous rate at which y is changing with respect to x. We can see this as the slope of the curve.
- Slope Formulas (Note the difference between $\frac{\Delta y}{\Delta x}$ and $\frac{dy}{dx}$)
  - The slope between two points $m = \frac{\Delta y}{\Delta x}$; $m = \frac{y_2 - y_1}{x_2 - x_1}$; $m = \frac{f(x+h) - f(x)}{h}$
  - Instantaneous slope $m = \frac{dy}{dx}$; $m = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$
- Example: Sheet#5, pr#4d Given $f(x) = x^3 + x^2 - 6x$

  $f'(x) = 3x^2 + 2x - 6$

  $f(-1) = 6$ This tells us that when x = -1, the y-value is 6.

  $f'(-1) = -5$ This tells us that when x = -1, the slope is -5.

  Key question: Where are the peaks and valleys of the curve f(x)?

  Answer: Where $f'(x) = 0$. $\therefore$ $f'(x) = 0 = 3x^2 + 2x - 6$,

  which yields answers (-1.786, 8.21) and (1.12, -4.06)

Day #12

Review:

- On the previous sheet, we came to the realization that the integral (which is geometric in nature) and the derivative (which is an algebraic process) are closely related; they are inverses of each other. Thus, the integral and the anti-derivative, F(x), are essentially equal.
- We also learned the power rule for integrals:

  If $f(x) = k \cdot x^n$ then $\int_0^a f(x)\,dx = k \cdot \frac{x^{n+1}}{n+1}$

- Example of how we can easily calculate the area under a curve:

  Question: What is the area under $f(x) = \frac{1}{3}x^4$ from x = 0 to 2?

  Solution: $F(x) = \frac{1}{3}\frac{x^5}{5} = \frac{x^5}{15}$. Therefore the area is $F(2) = \frac{32}{15}$.

- Review problems. These should be done on the board, and shouldn't take much time.
  - For each f(x), find f '(x) and find F(x).

    $f(x) = x^{13}$ $f(x) = 3x^5$ $f(x) = 7x$ $f(x) = 8$ $f(x) = x^9 + 10x^2 - 6x - 4$

Slope of a Line Tangent to a Curve.

- Review slope formulas from yesterday.
- Peaks and Valleys. Review ideas from yesterday, especially this key idea:

  In order to find a peak (local maximum) **or a valley** (local minimum) **we ask ourselves: "Where is the slope equal to zero?" This question is written $f'(x) = 0$.**

- Mention that the slope of a secant line connecting two points on a curve is calculated by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, which can also be written as $m = \frac{f(x+h) - f(x)}{h}$. The whole idea is the same as what we did for instantaneous speed.
  - Example: Given the two points (2,4) and (3,9) on the curve f(x) = x2, find the slope of the line connecting these two points.
  - What happens to the line, and its slope, if we "slide" the lower point up closer and closer to the upper point?
- Important Questions to understand:
  - Using the case of distance and speed…given $d(t) = 3t^2$ and $d'(t) = 6t$…
    - What does d(2) = 12 mean?
    - What does d'(5) = 30 mean?
  - Now using the more general case of $f(x) = 3x^2$ and $f'(x) = 6x$…
    - What does f(5) = 75 mean?
    - What does f'(5) = 30 mean?
- Do this Example: Find the slope of the curve $f(x) = -x^2 + 4x$ at x = 3.

- Do **Discovery Sheet #10 (*Tangents & Areas, Part I*)**

Day #13

Review:

- Go over the last problem from sheet #10, where it is necessary to use the derivative to find the exact local max and min.
- How do we calculate the area under the curve from x = a to b? The first step is to determine the anti-derivative. Then we evaluate F(x) at both a and b, and subtract.

Fundamental Theorem of Calculus.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

In words, this means that in order to find the area under the curve, we simply need to find the anti-derivative, evaluate it at the two x-values, and subtract.

- **Do Discovery Sheet #11 (*Tangents & Areas, Part II*)**
- Prepare for tomorrow's "Volume of Ellipsoid" by graphing it. And then giving the basic strategy.

Day #14

Review:

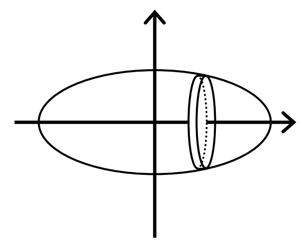
- Fundamental Theorem of Calculus.

Calculus History, Part II. (***TBC)

Volume of an Ellipsoid (or maybe instead derive the formula for the volume of a sphere.)

- Given the equation of an ellipse $\frac{y^2}{4} + \frac{x^2}{25} = 1$. Solving this for y gives us: $y = \pm\sqrt{4 - \frac{4}{25}x^2}$

  The top half of the ellipse is then $y = \sqrt{4 - \frac{4}{25}x^2}$
- If we spin this about the x-axis we get an ellipsoid, which can be considered as the sum of a infinite series of circular disks (i.e., thin cylinders) that are all perpendicular to the x-axis.



- The base of each disk is a circle that has an area equal to $\pi \cdot y^2$, which we will write as $g(x) = 4\pi - \frac{4\pi}{25}x^2$
- The volume of each disk (i.e., cylinder) is then $V = A_{base} \cdot H = g(x) \cdot \Delta x$
- The volume of the whole ellipsoid is therefore:

$$\lim_{n\to\infty} \sum_{k=1}^{n} g(x_k) \cdot \Delta x = \int_{-5}^{+5} g(x)dx = G(5) - G(-5)$$

Where $G(x) = 4\pi x - \frac{4\pi}{25} \cdot \frac{x^3}{3}$

$$G(5) = 20\pi - \frac{20\pi}{3} = \frac{40\pi}{3}$$

$$G(-5) = -20\pi + \frac{20\pi}{3} = -\frac{40\pi}{3}$$

$$\therefore \text{ Volume of Ellipsoid} = G(5) - G(-5) = \frac{40\pi}{3} - (-\frac{40\pi}{3}) = \frac{80\pi}{3}$$

- **Give Review Sheet**, to be finished as homework.

Day #15 & 16

Catch-up, review, and final test!

Calculus History (Should be done at some point in the final week)

- Historical Background. Give an overview of the key mathematical events in the 17th century. This shows how the 17th century was the "mathematical revolution". Huge progress in just 80 years.
- Galileo Galilei (1604): Uses experimentation to gather empirical data and thereby discovers his "Law of Falling Bodies".
- Johannes Kepler
  - 1596: In 1596 he published the book, *Mysterium Cosmographicum* in which he stated his hypothesis on the movement of the planets. It was based on the idea that the Platonic solids could be nested inside one another, and thereby defined the spacing of the planets. In short, the order was: sphere (Mercury), octahedron, sphere (Venus), icosahedron, sphere (Earth), dodecahedron, sphere (Mars), tetrahedron, sphere (Jupiter), cube, sphere (Saturn). After obtaining Tycho Brahe, he realized his theory was wrong (but very close!).

- (1610 & 1619): Publishes his three laws of planetary motion:

1. The orbits of every planets is an ellipse with the sun at one of the focal points.
2. The line joining any planet to the sun sweeps out equal area in equal time.
3. The squares of the revolutionary periods of any two planets is directly proportional to the cubes of their mean distance from the sun.

   Example: since Mars is 1.52 times as far from the sun as the earth is, this means that it's period (P) can be found from: P^2:(1 year)^2 = $(1.52)^3$:$(1AU)^3$, yielding an orbit of $1.52^{1.5} \approx 1.87$ years.

   Example: Neptune's period is about 13.9 times as long as Jupiter's. Therefore, Neptune must be $13.9^{2/3} \approx 5.781$ times as far from the sun as is Jupiter.

- René Descartes (1637): Publishes *Discourse on the Method*, which changes how we do science. The world is a machine! Science should be empirical and objective – based upon experiment.
- Descartes and Pierre de Fermat (1637 & 1635): First steps toward analytical geometry. Descartes: "We may give any value we please to either x or y and find the value of the other from this equation…"
- Blaise Pascal and Gerard Desargues (1640): First developments of projective geometry, although it was not recognized that it was a new type of geometry. Further development waited nearly 200 years.
- 17^{th} Century Puzzles. The calculus was created was primarily created to deal with the major math/science problems of the 17^{th} century. There were four problems of particular interest, each of which becomes easily solved with calculus:

  1. Instantaneous Speed. If a formula was known for the distance covered by a moving object, then what is the formula for its velocity and acceleration?
  2. Slope of the Tangent. Finding the slope of the tangent line to a curve. The solution to this was also important for scientific applications as well (e.g., design of lenses).
  3. Max and Min. How can we find the maximum and minimum values of a function?
  4. Measurement of Curves. This includes the length of curves, the area bounded by curves, and the volume of solids with curved surfaces.

- Sir Isaac Newton (born Xmas day 1642, the year Galileo died)
  - August 1665- Feb (?) 1667: Due to the plague, Newton (age 23) is forced to live in isolation for 18 months with his grandmother at the farm where he was born, in Woolsthorpe-by-Colsterworth, a hamlet in Lincolnshire, 100 miles north of London. Newton forms his basic ideas for differential calculus, and many laws of physics.
  - 1687: Newton publishes his most important book, *Philosophiae Naturalis Principia Mathematica*, which is a foundation for both classical mechanics and celestial mechanics. This includes his Universal Law of Gravitation. This law is able to unit Galileo's Law of Falling Bodies with Kepler's Law of Planetary Motion under one theory. But Newton is unsatisfied because that he has no explanation for what gravity is. "*I have not yet been able to discover the cause of these properties of gravity from phenomena and I feign no hypotheses...That one body may act upon another at a distance through a vacuum is to me a great absurdity.*"

    $$\boxed{F = G\,\frac{m_1 \cdot m_2}{r^2}}$$

  - 1693&1704: Newton did not publish his work on calculus because he feared he would be ridiculed for it. Finally, in 1693 he publishes part of an explanation of his calculus, and then 11 years later (1704) he publishes a fuller account of it. This is nearly 40 years after he developed his ideas.
- Gottfried Leibniz (1684): Publishes his comprehensive book on calculus.
- Leibniz had traveled in England in 1673, and possibly had then heard of some of Newton's thoughts on calculus at that time.
- In 1684, Leibniz publishes his comprehensive book on calculus.
- For much of the last 20 years of Newton's life (he died at age 84), he was bitterly engaged in a dispute with Leibniz over the authorship of calculus. He used his influential position as president of the Royal Society (1703 to his death) to slander Leibniz and accuse him of plagiarism, which was quite devastating to Leibniz, and ended up isolating the British mathematical community from the continental mathematicians, much to the detriment of the British.
- Marquis de L'Hospital (student of Leibniz) writes the first calculus textbook in 1696, which includes an axiom stating that two quantities differing by an infinitesimal can be considered equal.
- In the meantime, several mathematicians attacked the whole idea of calculus; they felt that it was "reckless".
  - The British mathematician, Bishop George Berkeley, attacked Newton's calculus (in 1734) saying famously, "…What are these fluxions? Zero, or not zero? May we not call them the ghosts of departed quantities?"
  - Yet, progress with calculus, and other fields based upon calculus continued quickly.
  - In 1754, Jean la Rond D'Alembert (French) realized that a mathematically sound theory of limits was needed for calculus.

- In 1821, Augustin-Louis Cauchy (French) finally developed an acceptable theory of limits, which then was refined over the remainder of the 19^{th} century.